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CONTENTS

Summer Conference in College Mathematics . . . B. W. Jones 75
Summer Conferences in 1954 76
The So-called Euler-Diderot Incident . . . . R. J. Gillings 77
The Formula for Change in Variables in a Multiple Integral . . . J. Schwartz 81
What the Colleges are Doing about the Poorly Prepared Student . W. L. Williams 86
On Numbers with Integral Harmonic Mean . Mariano Garcia 89
Geometrical Properties of Two-Dimensional Wave Motion . . . L. A. MacColl 96
Mathematical Notes . . . . . H. F. Sandham, L. Carlitz, S. Stein, David McLean 104
Classroom Notes . . . . . David B. Dekker, W. P. Reid, S. Parameswaran, Eugene Leimanis, R. C. Yates, M. R. Spiegel, Israel Halperin 113
Elementary Problems and Solutions 123
Advanced Problems and Solutions 126
Recent Publications 131
News and Notices 136
The Mathematical Association of America 145
New Members 145
October Meeting of the Minnesota Section 146
October Meeting of the Oklahoma Section 148
Calendar of Future Meetings 150

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Carl B. Allendoerfer, *Editor*

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SUMMER CONFERENCE IN COLLEGE MATHEMATICS

B. W. JONES, University of Colorado

The National Science Foundation supported a Summer Conference in Collegiate Mathematics which was held at the University of Colorado from June 15 to August 8, 1953. This was part of an effort to assist colleges and universities at some distance from the big research centers of the nation in their training of future mathematicians. The object of the Conference was not to stimulate research activity on the part of college teachers, but to cultivate through them greater interest and competence in mathematics on the part of their students—this to be done by means of the following auxiliary objectives:

1. To introduce recent mathematical ideas, beginning at the level of understanding of its members, encouraging them by study and by talking mathematics with their fellows to acquire an appreciation of present modes of thought and development.

2. To show and give opportunities for discussion of devices (*e.g.*, comprehensive examinations, honors programs, senior theses, *etc.*) for stimulating undergraduate activity and achieving competence.

3. To revivify old ideas in terms of modern thought.

The two full-time lecturers were Professors Emil Artin of Princeton University and Raymond L. Wilder of the University of Michigan. The former in his lectures developed the fundamental notions of algebra beginning with set theory, groups, rings, fields, vector spaces, valuation theory, finally leading up to the foundations of algebraic geometry. During the question period there were from time to time discussions on curriculum as well as on the subject of the lectures.

The material offered by Professor Wilder under the title "Foundations of Analysis and Geometry" was divided into two parts:

1. Introduction to modern foundational and topological concepts via the notion of curve.

2. "Gap-filling courses"; courses designed to bridge the gap between the elementary courses and the courses in function theory, modern analysis, applied mathematics, *etc.*

Part-time lecturers were Professor Pólya of Stanford University whose lectures bore the title "Great and Small Examples of Problem Solving," Professor E. P. Northrop of the University of Chicago, and Dr. Carroll V. Newsom, Associate Commissioner of Education of the State of New York, who discussed curriculum. There were many formal and informal discussions about mathematics and problems connected with college teaching.

There were about 80 members of the Conference. Eight came from New England, 24 from the rest of the Atlantic seaboard, 16 between the Appalachians and the Mississippi, and 33 from west of the Mississippi. Some members received some assistance from grants included in the plan, others from their own institutions, and many came entirely at their own expense. They were for the

most part housed in one of the residence halls of the University and thus had the opportunity of becoming well acquainted and comparing notes at all times of the day.

Notes of the lectures were prepared for the members by assistants assigned to each lecturer; Dr. Robert R. Stoll of Oberlin College for Professor Artin, Dr. Dan E. Christie of Bowdoin College for Professor Wilder, Dr. Chester Luther of Willamette College for Professor Pólya, Dr. Lloyd Williams of Reed College for Professor Northrop, and Father William C. Doyle of Rockhurst College and Dr. Robert Gordon of Hampton Institute for Dr. Newsom.

The members of the Conference were admitted to all events of the summer recreation program of the University and the city which they wished to attend. They had full library privileges, and a room in the library was assigned for their exclusive use.

The total effect of the Conference will be very hard to assess, but certainly the members worked very hard and conscientiously, were enthusiastic about the Conference and were very alive to the impact of the Conference on their own teaching.

SUMMER CONFERENCES IN 1954

In view of the success of the 1953 Summer Conference at Boulder, the National Science Foundation is sponsoring two similar conferences for college teachers of mathematics during the summer of 1954.

In the East there will be a conference at the University of North Carolina, Chapel Hill, under the direction of Professor E. A. Cameron. The conference will run from June 15 to August 7. The principal speakers will be Professor Tibor Rado of Ohio State University and Professor Emil Artin of Princeton University.

In the West there will be a conference at the University of Oregon, Eugene, under the direction of Professor Ivan Niven. The conference will run from June 21 to August 13. The principal speakers will be Professor Hans Rademacher of the University of Pennsylvania, and Professor D. G. Bourgin of the University of Illinois.

Detailed information concerning these conferences may be obtained from their respective directors.

THE SO-CALLED EULER-DIDEROT INCIDENT

R. J. GILLINGS, Sydney University, Australia

1. Introductory Notes. Peter I or Peter the Great, the first emperor of Russia, died in 1725 and was succeeded by his wife Catherine I. She reigned only two years, being followed by Peter II who died in 1730. Then came in relatively quick succession, Anne, daughter of Peter's half-brother, 1730–40; Ivan VI, grand-nephew of Anne, 1740–41; Elizabeth, daughter of Peter and Catherine I, 1741–61; Peter III, grandson of Peter, 1761–62; and Catherine II who reigned the 34 years from 1762 to 1796.

Catherine II is by some historians referred to as Catherine the Great, while others deny her the right to that title. However she was a woman of culture who respected the arts generally, encouraged men of letters, and who herself displayed ability as a writer.

Denis Diderot (1713–84) was a distinguished philosopher, an encyclopaedist, and an author of many scientific publications, who visited St. Petersburg in 1773–74 since the Empress Catherine II had purchased his library and appointed him its first librarian. The following mathematical memoirs of Diderot published in 1784, among others, are discussed by L. G. Krakuer and R. L. Krueger, [1]: "Sur la tension des cordes," "De la developpante du cercle," "Résistance de l'air au mouvements des pendules"; and in J. L. Coolidge's, *The Mathematics of Great Amateurs*, Oxford, 1949, ch. XIV, Diderot's treatment of "vibrating strings" is discussed, together with those mentioned above. Diderot, who had been in turn, a deist, a pantheist, a sceptic and finally an atheist, was then verging on 60 years of age, Catherine was 44 and Euler was 66.

Leonard Euler (1707–83), famous mathematician, was invited to St. Petersburg in 1727, to accept the chair of mathematics at the Academy recently founded by Peter the Great. Here he remained until 1741, when he was induced by Frederic the Great of Prussia to spend 25 years in Berlin until 1766, when he returned to St. Petersburg at the request of Catherine II, and worked there until the end of his life.

Dieudonné Thiébault (1733–1807) was a French man of letters who for the period from 1765 to 1784, lived in Berlin at the invitation of Frederic the Great. He was a member of the Berlin Royal Academy. The incident to which this paper refers had its origin in Thiébault's reminiscences of his twenty years sojourn in Berlin.

2. De Morgan's Statement. [2]. "The following anecdote is found in Thiébault's *Souvenirs de vingt ans de séjour à Berlin*, published in 1804. Thiébault does not claim personal knowledge of the anecdote, but he vouches for its being received as true all over the north of Europe. Diderot paid a visit to Russia at the invitation of Catherine the Second. At that time he was an atheist or at least talked atheism: it would be easy to prove him one thing or the other from his writings. His lively sallies on this subject much amused the Empress,

and all the younger part of her Court. But some of the older courtiers suggested that it was hardly prudent to allow such unreserved exhibitions. The Empress thought so too, but did not like to muzzle her guest by an express prohibition: so a plot was contrived. The scorner was informed that an eminent mathematician had an algebraical proof of the existence of God, which he would communicate before the whole Court, if agreeable. Diderot gladly consented. The mathematician, who is not named, was Euler. He came to Diderot with the gravest air, and in a tone of perfect conviction said, '*Monsieur*! $(a+b^n)/n=x$, *donc Dieu existe*; *répondez*!' Diderot, to whom algebra was Hebrew, though this is expressed in a very roundabout way by Thiébault—and whom we may suppose to have expected some verbal argument of alleged algebraical closeness, was disconcerted, while peals of laughter sounded on all sides. Next day he asked permission to return to France, which was granted."

3. Cajori's Statement. [3]. "The story goes that when the French philosopher Denis Diderot paid a visit to the Russian Court, he conversed very freely and gave the younger members of the Court circle a good deal of lively atheism. Thereupon Diderot was informed that a learned mathematician was in possession of an algebraical demonstration of the existence of God, and would give it to him before all Court, if he desired to hear it. Diderot consented. Then Euler advanced towards Diderot, and said gravely, and in a tone of perfect conviction: '*Monsieur*, $(a+b^n)/n=x$, *donc Dieu existe*; *répondez*!' Diderot, to whom algebra was Hebrew, was embarrassed and disconcerted, while peals of laughter rose on all sides. He asked permission to return to France at once, which was granted." [A reference is then given to De Morgan's *Budget of Paradoxes*.]

4. Bell's Statement. [4]. "We shall tell once more the famous story of Euler and the atheistic (or perhaps only pantheistic) French philosopher Denis Diderot (1713–1784). Invited by Catherine the Great to visit her Court, Diderot earned his keep by trying to convert the courtiers to atheism. Fed up, Catherine commissioned Euler to muzzle the windy philosopher. This was easy because all mathematics was Chinese to Diderot. De Morgan tells what happened (in his classic *Budget of Paradoxes*, 1872): Diderot was informed that a learned mathematician was in possession of an algebraical demonstration of the existence of God, and would give it before all the Court, if he desired to hear it. Diderot gladly consented . . . Euler advanced towards Diderot, and said gravely, and in a tone of perfect conviction: '*Sir*, $(a+b^n)/n=x$, *hence God exists*; *reply*!' It sounded like sense to Diderot. Humiliated by the unrestrained laughter which greeted his embarrassed silence, the poor man asked Catherine's permission to return at once to France. She graciously gave it. Not content with this masterpiece, Euler in all seriousness painted his lily with solemn proofs, in deadly earnest, that God exists and that the soul is not a material substance. It is reported that both proofs passed into the treatises on theology of his day."

5. Hogben's Statement. [5]. "There is a story about Diderot, the Encyclo-

paedist, and materialist, a foremost figure in the intellectual awakening which immediately preceded the French Revolution. Diderot was staying at the Russian court, where his elegant flippancy was entertaining the nobility. Fearing that the faith of her retainers was at stake, the Tsaritsa commissioned Euler, the most distinguished mathematician of the time, to debate with Diderot in public. Diderot was informed that a mathematician has established a proof of the existence of God. He was summoned to court without being told the name of his opponent. Before the assembled court, Euler accosted him with the following pronouncement, which was uttered with due gravity: '$(a+b^n)/n=x$, *donc Dieu existe, répondez*!' Algebra was Arabic to Diderot. Unfortunately he did not realise that was the trouble. . . . Translated freely into English it may be rendered: 'A number x can be got by first adding a number a to a number b multiplied by itself a certain number of times, and then dividing the whole by the number of b's multiplied together. . . .' Like many of us Diderot had stage-fright when confronted by a sentence in size language. He left the court abruptly amid the titters of the assembly, confined himself to his chambers, demanded a safe conduct, and promptly returned to France."

6. Thiébault's Statement. [6]. An English translation of the relevant passage is as follows: From the moment of his arrival Diderot was well received, all his expenses had been paid by the Empress whom he amused immensely by the fecundity and fire of his imagination, by the abundance and singularity of his ideas, and by the zeal, boldness and eloquence with which he publicly upheld atheism. But several of the older courtiers more experienced and perhaps more easily alarmed, persuaded their autocratic sovereign that teachings of this kind could have unfortunate consequences for the whole court, and especially among the large youthful group, destined for important empire posts, who might embrace this doctrine with more eagerness than careful scrutiny. The Empress then desired that some restraint be put upon Diderot on this subject, provided that she did not appear to play any part in the matter, and provided that no one should show any undue authority about it. It was therefore announced to the French philosopher one evening, that a Russian philosopher, a learned mathematician and a distinguished member of the Academy, was prepared to prove the existence of God to him, algebraically, and before the whole court. Diderot said that he would be happy to listen to such a demonstration, in the validity of which of course, he did not believe, and so an hour and a day were fixed to convince him. The occasion having arrived, with the whole court present, that is to say, the men and more particularly the younger members, the Russian philosopher gravely advanced towards the French philosopher, and speaking in a tone of voice to imply his full conviction, said, "*Monsieur*, $(a+b^n)/z=x$, [7] *therefore God exists*: *answer that*!" Diderot was willing to show the futility and stupidity of this so-called proof, but felt in spite of himself, the embarrassment that one would, on discovering, (among them), their intention of making a game of it, so that he was not disposed to attempt to admonish them

for the indignities proposed for him. This adventure made him fearful that there might be others in store for him of a like nature, and so sometime afterwards he expressed his desire to return to France. Then the Empress having declared her willingness to pay all his traveling expenses, he was sent on his journey after having received 50,000 francs. Eventually his carriage was wrecked near Riga, but he received from the governor of that town the whole of the cost of the repairs. I do not assert the truth of any one of these facts; I say only, that at that time, they were talked about, and were believed by the inhabitants of the north.

7. Conclusions. Since Thiébault's statement is the only authority for the facts discussed in this incident we may now summarize some of the unwarranted changes made by the authors mentioned. We see that De Morgan's statement makes more than one departure from his quoted authority: "Algebra was Hebrew" . . . (see also the facts given in the Introductory Notes), "Diderot . . . was disconcerted while peals of laughter sounded on all sides," "Next day he asked permission to return to France," and "The mathematician who was not named was Euler." We grant that Thiébault's phrase, "a Russian philosopher, a learned mathematician and a distinguished member of the academy," seems rather definitely to refer to the Swiss Mathematician Euler.

De Morgan's inventions are naturally repeated by Cajori. Bell also repeats them but substitutes "All mathematics was Chinese to Diderot," for "Algebra was Hebrew to Diderot." No authority is given for Bell's statements in his final two sentences, hence we question the adequacy of that authority.

It will be observed that Hogben also alters the form of De Morgan's inventions. Struik has well pointed out [8] the incongruity of the story, which is completely out of character both for the devout Euler and the highly intelligent Diderot. Thiébault himself was not convinced of the truth of it. The extent to which legendary stories of history may be distorted is well illustrated by the so-called Euler-Diderot incident.

References

1. L. G. Krakuer and R. L. Krueger, *Isis*, ol. 33, 1941, p. 219–231.

2. A. De Morgan, *A Budget of Paradoxes*. London, 1872, p. 250 and p. 474. The story appears twice.

3. F. Cajori, *A History of Mathematics*. New York, 1919, p. 233. Second edition revised and enlarged.

4. E. T. Bell, *Men of Mathematics*. New York, 1937, p. 146–147.

5. L. Hogben, *Mathematics for the Millions*. New York, 1951, p. 17. Opening paragraph of chapter I.

6. D. Thiébault, *Mes Souvenirs de Vingt Ans de Séjour à Berlin*. Paris, 1804, 3 vols. The passage is on page 141 of volume 3. The English translation published in 1806 in 2 volumes at Philadelphia, under the title of, *Anecdotes of Frederick the Great of Prussia*, makes no mention of this story.

7. The formula is printed in italics and the *s* of the denominator could possibly be mistaken for a 2. It is however "*z*" clearly enough and not "*n*" as De Morgan miscopied it and following him, Cajori, Bell, Smith, Sanford, Hogben and others.

8. D. J. Struik, *A Concise History of Mathematics*. New York, 1948, p. 182.

THE FORMULA FOR CHANGE IN VARIABLES IN A MULTIPLE INTEGRAL

J. SCHWARTZ, Yale University

A well-known theorem of elementary analysis states:

THEOREM. *Let D_1 and D_2 be open sets in Euclidean n-space E^n, and let h: $D_1 \to D_2$ be a* 1-1 *mapping of D_1 onto D_2 such that h and its inverse h^{-1} are continuous and have continuous derivatives. Let*

$$J(x) = \left| \det \left(\frac{\partial h^i(x)}{\partial x^j} \right) \right|$$

be the absolute value of the Jacobian determinant of the transformation h. Then a function $f(x)$ is integrable over the domain D if and only if the function $f(h(x))J(x)$ is integrable over D_1, and we have

$$\int_{D_2} f(x)dx = \int_{D_1} f[h(x)]J(x)dx.^* \tag{1}$$

Before going over to the formal proof, we shall make some historical remarks. It is easy to see, from the formal multiplicative property of Jacobian determinants, that if qur theorem holds for each of two transformations, it holds for their product. All but one of the known proofs make use of this fact, in a greater or lesser degree. The most naive idea would be to divide D_1 into "infinitesimal" rectangular parallelepipeds (shortly: rectangles), to observe that the image by h of such a rectangle is an infinitesimal parallelepiped, to compute the volume of this parallelepiped, to sum all the infinitesimals arising in this way, and thus to arrive at formula (1). Even this proof, however, makes use of the determinant formula for the volume of a parallelepiped in E^n; and this determinant formula can only be proved by an inductive argument which amounts, in essence, to use of the "multiplicative" principle enunciated above.

Moreover, to turn this heuristic approach into a rigorous proof is not entirely trivial. It is done, to my knowledge, only in two places: Jourdan's *Cours de Analyse* and O. Haupt's very interesting three volume work on real variable theory, *Differential-und Integralrechnung unter besondere Berucksichtigung neuere Ergebnisse*. In each of these proofs a small rectangle C is considered, and, by careful construction, two parallelepipeds, the first entirely containing and the

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* Throughout this paper the symbol x stands for the vector $[x_1, x_2, \cdots, x_n]$. The integrals in equations like (1) are understood to be n-fold multiple integrals. For instance $\int f(x)\,dx = \int \cdots \int f(x_1, x_2, \cdots, x_n)dx_1dx_2 \cdots dx_n$.

second entirely contained within the image $h(C)$ of C are found. In this way, sufficiently exact bounds for the volume of $h(C)$ are found so that formula (1) can be obtained in the limit. The principal difficulty in this proof is that of finding the parallelepiped interior to $h(C)$ since the exterior parallelepiped can be established without difficulty. The formal proof to be given below is essentially a simplified version of this proof which makes use of a convenient device for avoiding the interior parallelogram.

Most authors have, however, preferred to avoid this line altogether, with the result that the most popular proof (Courant, Hobson, Pierpont, Franklin, Goursat, Gibson, Levy, Hadamard, Stewart, Graves, and Humbert are among those who reproduce it) is one which operates much more strongly with the multiplicative principle, than the proof just outlined. The proof is carried through by these authors by making the "change of variables" $x \to h(x)$ one variable at a time, and thus by reducing the theorem, via the multiplicative principle, to the one variable case. This brings to the fore another difficulty: the intermediate mappings need not be one-to-one "in the large," and hence the domain D_1 must be decomposed into subdomains to make the proof work. All in all, the obstructing details met with along this line of march are rather formidable, and the "non-inductive" proof described in the previous paragraph is consequently simpler in principle.

The third type of proof met in the literature (*c.f.* Widder, de la Vallée Poussin, Osgood) is a sort of dimensional induction which uses Green's theorem rather than the multiplicative principle. This method is, however, open to serious objection. First of all, Green's theorem in the form required cannot even be stated, much less proved, without having recourse to such notions as inside and outside of a Jordan curve, positive and negative orientation of a curve, *etc.*, and hence to the profound ideas involved in the algebraic topology of the plane and of E^n. Secondly, though this proof is "elementary" in the plane, its extension to E^n with $n > 2$ requires that use be made of a part of the formal machinery of exterior differentiation and tensor analysis, and thus the proof loses its elementary character.

These introductory remarks made, let us get to work. First of all, we note that by decomposing an arbitrary function $f(x)$ (which we can suppose continuous) into its positive and negative parts, we can see that it is sufficient to prove (1) when $f(x)$ is nowhere negative. If h is linear, the result then follows easily enough by the multiplicative principle. Indeed, an arbitrary linear transformation can be written as a product of linear transformations of three basic types:

$$\text{(a)} \qquad t_\lambda[x_1, x_2, \cdots, x_n] = [\lambda x_1, x_2, \cdots, x_n]$$

$$\text{(b)} \qquad a[x_1, x_2, \cdots, x_n] = [x_1 + x_2, x_2, \cdots, x_n]$$

$$\text{(c)} \qquad s_{ij}[x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n] = [x_1, \cdots, x_j, \cdots, x_i, \cdots, x_n].$$

We have, however,

$$|\det(t_\lambda)| \int_{E^n} f(t_\lambda x)dx = |\lambda| \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(\lambda x_1, x_2, \cdots, x_n)dx_1 \cdots dx_n$$

(a)
$$= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1, \cdots, x_n)dx_1 \cdots dx_n$$
$$= \int_{E^n} f(x)dx$$

by the representation of a multiple integral as a repeated integral (Fubini's theorem) and by the obvious one-variable result:

$$|\lambda| \int_{-\infty}^{+\infty} g(\lambda y)dy = \int_{-\infty}^{+\infty} g(y)dy.$$

(b)
$$\int_{E^n} f(ax)dx = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1 + x_2, x_2, \cdots, x_n)dx_1 \cdots dx_n$$
$$= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1, x_2, \cdots, x_n)dx_1 \cdots dx_n$$

by Fubini's theorem and the obvious one-variable result:

$$\int_{-\infty}^{+\infty} g(y + z)dy = \int_{-\infty}^{+\infty} g(y)dy.$$

Further

$$\int_{E^n} f(s_{ij}x)dx$$
$$= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1, \cdots, x_j, \cdots, x_i, \cdots, x_n)dx_1 \cdots dx_i \cdots dx_j \cdots dx_n$$

(c)
$$= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n)dx_1 \cdots dx_j \cdots dx_i \cdots dx_n$$
$$= \int_{E^n} f(x)dx$$

by Fubini's theorem, which tells us that a multiple integral is the same as the repeated integral in any order of integration.

These three facts give us the desired result for linear maps for functions defined and integrable in all of E^n. However, if $f(x)$ is defined and integrable in a domain D, we can extend its domain of definition to all of E^n by the trivial convention $f(y) = 0$ for y not in D, and thus our result on linear maps holds for an arbitrary domain D. What, however, about the most general maps described in the hypotheses of our theorem? To consider these in an effective way,

we first introduce some notational conventions. If $x \in E^n$, so that $x = [x, \cdots, x_n]$, we put $|x| = \max_{1 \leq i \leq n} |x_i|$. This "norm" has the convenient property that in terms of it, a cube with center p and side-length $2s$ can be characterized by the restriction $|x-p| \leq s$. If $A: E^n \to E^n$ is the linear transformation represented by the matrix a_{ij}, so that

$$A(x) = A[x_1, \cdots, x_n] = \left[\sum_{j=1}^{n} a_{1j}x_j, \cdots, \sum_{j=1}^{n} a_{nj}x_j \right],$$

we put

$$|A| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.$$

Thus, $|A(x)| \leq |A||x|$. We also introduce the Jacobian matrix $j(x) = j_{ik}(x)$ of the transformation h: $h(x) = [h_1(x) \cdots h_n(x)]$ by putting

$$j_{ik}(x) = \frac{\partial h_i(x)}{\partial x_k}.$$

Let us agree to write the volume of a set S in E^n as $\mu(S)$. Then if C is a cube in the open set D_1, so that C is the set of all x characterized by a condition of the form $|x-p| \leq s$, we have $\mu(C) = (2s)^n$. We have, by the mean value theorem,

$$h_i(x) - h_i(p) = \sum_{k=1}^{n} j_{ik}[p + \theta_i(x)(x-p)](x_k - p_k),$$

where $0 \leq \theta_i(x) \leq 1$. It follows immediately that

$$|h(x) - h(p)| \leq s \max_{y \in C} |j(y)|;$$

i.e., $h(C)$ is entirely contained in the cube defined by

$$|z - h(p)| \leq s \max_{y \in C} |j(y)|,$$

so we see that

$$\mu(h(C)) \leq \left\{ \max_{y \in C} |j(y)| \right\}^n \mu(C).$$

We apply this formula to the map $A^{-1}h$, where A is an arbitrary non-singular linear map, and make use of the fact that we can show that $\mu(A^{-1}(S)) = |\det(A^{-1})| \mu(S)$ for an arbitrary closed set S. To prove this we can apply our theorem in the special case when h is a linear map to the function f defined by $f(x) = 1$ for x in $A^{-1}(S)$, $f(x) = 0$ for x not in $A^{-1}(S)$. Let us apply this to $S = h(C)$. The set $h(C)$ is closed since C is closed and bounded and h is a

continuous map. Since

$$|\det(A^{-1})|\,\mu(h(C)) \leqq \left\{\max_{y\in C} |A^{-1}j(y)|\right\}^n \mu(C),$$

we obtain

$$\mu(h(C)) \leqq |\det(A)| \left\{\max_{y\in C} |A^{-1}j(y)|\right\}^n \mu(C). \tag{2}$$

Now, let the cube C be subdivided into a finite set $C_1 \cdots C_M$ of non-overlapping cubes with centers $x_1 \cdots x_M$, and suppose that δ is greater than the length of a side of any of them. Apply (2) to each of $C_1 \cdots C_M$, taking, however, $A = j(x_i)$ in applying (2) to C_i. Then add. This gives

$$\mu(h(C)) \leqq \sum_{i=1}^{M} |\det(j(x_i))| \left\{\max_{y\in C_i} |j^{-1}(x_i)j(y)|\right\}^n \mu(C_i).$$

Now, since $j(x)$ is a continuous (matrix-valued) function, we have $j^{-1}(z)j(y)$ approaching the identity matrix δ_{ij} as z approaches y, and hence we have

$$\left\{\max_{y\in C_i} |j^{-1}(x_i)j(y)|\right\}^n \leqq 1 + \eta(\delta),$$

where $\eta(\delta)$ approaches zero with δ. This gives

$$\mu(h(C)) \leqq [1 + \eta(\delta)] \sum_{i=1}^{M} |\det(j(x_i))|\,\mu(C_i);$$

as δ approaches zero, the sum on the right approaches $\int_C J(x)dx$, and the inequality becomes

$$\mu(h(C)) \leqq \int_C J(x)dx.$$

This leads immediately to the formula

$$\int_{D_2} f(x)dx \leqq \int_{D_1} f(h(x))J(x)dx \tag{3}$$

for all non-negative functions. Now, if we apply (3) to the inverse map h^{-1}, we find

$$\int_{D_2} f(x)dx \leqq \int_{D_1} f(h(x))J(x)dx \leqq \int_{D_2} f(h(h^{-1}(x)))J(h^{-1}(x))J(x)dx,$$

where $J'(x)$ is the absolute value of the Jacobian determinant of the map h^{-1}. Now, we have only to apply the multiplicative principle $J(h^{-1}(x))J'(x) = 1$, and our proof of (1) is complete.

WHAT THE COLLEGES ARE DOING ABOUT THE POORLY PREPARED STUDENT*

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1. Introduction. During the last World War when a Navy V-12 unit with students from all parts of the United States was stationed at the University of South Carolina, it became evident early in the program that many of these students were wholly unprepared for the usual beginner's courses in college mathematics even though a large number of them had had one year of plane geometry and from one to three years of algebra in high school.

To meet this situation remedial non-credit courses in both algebra and geometry were introduced in our curriculum with the intention of dropping them at the end of the war, since we thought that there was some relation between the poor performance of the V-12 trainees in mathematics and the tenseness of the times brought on by the war. However, when the war ended, we found that the problem remained, which seemed to indicate that the war merely served to bring to the surface and emphasize a condition which previously existed; namely, that the high schools in our area of the country were not preparing their graduates to enter our freshman mathematics courses even though these students thought they were prepared when they came to us.

Since we found it necessary to continue our remedial courses, I decided last fall to see to what extent the other institutions of higher learning in the United States were having the same problem and what they were doing about it. To accomplish this, something over five hundred questionnaires were sent out and replies were received from institutions in forty-six states with student bodies ranging in size from one hundred and fifty to nineteen thousand and with mathematics departments from ten to four thousand students. The study, therefore, was national in scope and embraced institutions from the smallest to the largest.

2. A lack of confidence revealed in high school mathematics units. The survey showed that forty-one per cent of the colleges and universities in the country no longer accept students for their freshman mathematics courses on the basis of their high school work in mathematics, but instead require a placement test before registration. The reason for this lack of confidence on the part of the colleges and universities is understandable in view of the fact that the questionnaire returns show that, of the students now taking college algebra in our institutions of higher learning, only twenty per cent are regarded as good and thirty-two per cent as fair, while twenty-nine per cent are regarded as poor and nineteen per cent as totally unprepared. The corresponding percentages for trigonometry are about the same, except that twenty-five per cent are regarded as totally unprepared for the course. A quicker picture of the situation may be

* Presented to the Southeastern Section of the Mathematical Association of America, Alabama Polytechnic Institute, March 13, 1953.

obtained by simply stating that in the opinion of those in charge of the various mathematics departments throughout the country about one-half of the students now trying to take college algebra and trigonometry are either poorly or totally unprepared. The problem becomes more serious when it is realized that the students involved in this study are just out of high school and not veterans to any great extent whose poor showing could, perhaps, be attributed in part to the interruption of their education by military service.

3. Remedial courses. Because of the faulty preparation in high school algebra and geometry of so many students now going to our colleges and universities a large number of these institutions are finding it necessary to give remedial courses in these subjects.

The situation with reference to algebra is very serious, for the survey showed that sixty-two per cent of the institutions of higher learning in the country are now giving remedial courses in this subject. The questionnaire also revealed that, of the students now coming to our colleges and universities who would normally take college algebra, thirty-four per cent are in the remedial algebra course. In other words, our institutions of higher learning are today teaching high school algebra (or its equivalent) to slightly more than one-third of their students who thought they were prepared for college algebra. The percentage of unprepared students for college algebra is much larger than the number in the remedial course would indicate, since many students do not take the remedial course. They either drop out of college during the first few days or change to some curriculum not requiring mathematics.

Seventeen per cent of the colleges and universities are giving remedial courses in geometry, but of those giving this course only thirty-three per cent allow credit. In the algebra remedial course sixty-one per cent allow credit.

4. How the problem is being met. As has already been pointed out a large percentage of the colleges and universities are meeting the problem of the unprepared student for college mathematics by giving remedial courses at the high school level. Some of the institutions which replied to the questionnaire stated that they do not admit students to their mathematics departments if they are deficient in high school mathematics. Those which do admit them and are not giving remedial courses are trying to solve the problem in numerous ways among which are the following: Students are required to make up their deficiencies in night school or through the institution's extension division; tutors are provided with the students paying for this service; a more elementary algebra course is given and is made a terminal course; poorly prepared students are advised to take a course of study which does not require mathematics. A number of the institutions frankly admitted that they are doing nothing about the matter—are letting these problem students "sink or swim," with the implication that few of them swim.

5. A suggested solution. Even though a large percentage of the colleges and

universities are giving remedial courses as a means of meeting the problem of the unprepared student in mathematics this is undoubtedly not the best solution. Our institutions of higher learning should not be teaching high school subjects. It is usually cheaper for the high schools to do it and the chances are that they can do it better. I am fully aware of the problem which the colleges and universities face. The problem is a real one. When the unprepared student comes to our campuses with a high school diploma and the usual units in mathematics he has a right to think that he is ready for college. If he is not, he can be denied admission or accepted and helped to overcome his deficiencies. The remedial courses are being used for this purpose. But their use should be regarded as only a temporary solution until one of a permanent nature can be found. This study tried to ascertain the thinking of the colleges and universities in this direction.

Of those replying to the questionnaire from the forty-six states covered, eighty-eight per cent favored a more fundamental drill on the basic ideas and operations in algebra (with no attempt to treat such topics as progressions, complex numbers, *etc.*, which are given in college algebra), together with a thorough review of algebra in the senior year. Only twelve per cent were in favor of trying to meet the problem of the poorly prepared student for college mathematics by giving more algebra in high school. These percentages are most significant. They show unmistakably that the overwhelming majority of the institutions of higher learning believe that the solution of the problem of the poorly prepared student for college mathematics could be solved by the high schools covering less material in their high school mathematics courses and covering it more thoroughly.

To bring about this revision and to, perhaps, accomplish other desirable ends, there must be a closer relationship between our colleges and universities and our secondary schools in order that our mutual problems can be better understood. One way to achieve this end is for those of us in our colleges and universities to cooperate with our secondary schools through the various state education associations and to actively participate in the programs sponsored by these associations.

Recently in South Carolina through the cooperation of some of our high school, elementary and college teachers, machinery was set in motion looking towards the preparation of a syllabus under the direction of a committee composed of representatives of the above group to be used in all mathematics classes from the elementary grades through the senior year in high school. We believe that this syllabus will improve the teaching of mathematics at all levels in our secondary schools and will ultimately result in these schools sending to our colleges and universities graduates who are prepared for college mathematics.

ON NUMBERS WITH INTEGRAL HARMONIC MEAN

MARIANO GARCIA, University of Puerto Rico

The results given in this paper were developed as part of an undergraduate seminar held under my supervision at the University of Puerto Rico College of Agriculture and Mechanic Arts during the second semester of the year 1951–52. We have obtained all the positive integers less than 10,000,000 which have an integral harmonic mean for their divisors, thus extending the table given by Ore [1, p. 618] of the integers between 1 and 10,000 having the same property. The method used to obtain these numbers will be illustrated for some cases only, as the computations involved all follow a similar pattern and are somewhat long in a few cases. There are some theorems which are helpful in reducing the amount of computation. Before stating these we shall make some preliminary remarks.

It will be recalled [1, p. 615] that the harmonic mean $H(n)$ of the divisors of a number n is defined by the formula

$$\frac{1}{H(n)} = \frac{1}{v(n)} \cdot \sum_{d|n} \frac{1}{d}$$

where d stands for a divisor of n and $v(n)$ denotes the number of divisors of n. From this it follows that

$$H(n) = \frac{nv(n)}{\sigma(n)}$$

where $\sigma(n)$ denotes the sum of the divisors of n. We also have that if

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$$

is the decomposition into prime factors of the number n,

$$v(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1)$$

and

$$\sigma(n) = \sum_{d|n} d = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{\alpha_2+1} - 1}{p_2 - 1} \cdots \frac{p_r^{\alpha_r+1} - 1}{p_r - 1}.$$

From these formulas it follows that

$$H(a \cdot b) = H(a) \cdot H(b)$$

when a and b are relatively prime.

THEOREM 1 (Ore) [1, p. 616]. *A power of a prime can not have an integral harmonic mean.*

THEOREM 2. *If n is odd and has an integral harmonic mean, the prime factor decomposition of n can not contain a prime of the form $(4k-1)$ raised to an odd power.*

Proof. This result follows from the fact that if p is a prime of the form $s\cdot 2^m-1$ and $\alpha+1=2^v q$, where s, m, v and q are positive integers, s and q odd, then $\sigma(p^\alpha)=2^r t$ where t is an odd integer and $r=v+m-1$.

Thus, if an odd integer n contains a prime of the form $(4k-1)$ raised to an odd power, $\sigma(n)$ will contain 2 as a factor at least once more than $v(n)$ will and consequently $H(n)=(nv(n)/\sigma(n))$ can not be integral.

THEOREM 3. *No odd number having a prime factor decomposition of the form $p_1^{2^{s_1}-1}p_2^{2^{s_2}-1}\cdots p_r^{2^{s_r}-1}$ can have an integral harmonic mean.* (Note that this theorem is a generalization, for odd numbers, of a theorem due to Ore [1, top p. 617].)

Proof. If $n=p_1^{2^{s_1}-1}p_2^{2^{s_2}-1}\cdots p_r^{2^{s_r}-1}$, we may assume $p_1<p_2<\cdots<p_r$. Since

$$H(n)=\frac{p_1^{2^{s_1}-1}\,p_2^{2^{s_2}-1}\cdots p_r^{2^{s_r}-1}}{\frac{p_1+1}{2}\cdot\frac{p_1^2+1}{2}\cdots\frac{p_1^{2^{s_1}-1}+1}{2}\cdots\frac{p_r+1}{2}\cdot\frac{p_r^2+1}{2}\cdots\frac{p_r^{2^{s_r}-1}+1}{2}},$$

$(p_1+1)/2$ can not cancel and $H(n)$ can not be integral.

THEOREM 4. *Except for perfect numbers, no integer with prime factor decomposition $p_1^\alpha p_2$ can have an integral harmonic mean.*

Proof. Let n have the prime factor decomposition $p_1^\alpha p_2$ and suppose that

$$H(n)=\frac{p_1^\alpha\cdot p_2\cdot(\alpha+1)\cdot 2}{(p_1^\alpha+\cdots+p_1+1)(p_2+1)}$$

is integral.

Case i. $p_2=2$.

$$H(n)=\frac{p_1^\alpha\cdot 2^2\cdot(\alpha+1)}{(p_1^\alpha+\cdots+p_1+1)\cdot 3}.$$

Thus $p_1^\alpha+\cdots+p_1+1$ must be a factor of $4(\alpha+1)$. But for $\alpha>1$, $p_1^\alpha+\cdots+p_1+1\geqq 3^\alpha+\cdots+3+1>4(\alpha+1)$. Therefore $\alpha=1$, and

$$H(n)=\frac{p_1\cdot 2^2}{(p_1+1)\cdot 3}.$$

Consequently $p_1=3$ and $n=6$, a perfect number.

Case ii. $p_2 \neq 2$.

$$H(n) = \frac{p_1^{\alpha} \cdot p_2 \cdot (\alpha + 1)}{(p_1^{\alpha} + \cdots + 1)\dfrac{p_2 + 1}{2}}.$$

If $\alpha+1$ is prime, since $\alpha+1 < p_1^{\alpha} + \cdots + 1$, $p_1^{\alpha} + \cdots + 1$ must have p_2 as a factor. Thus $p_1^{\alpha} + \cdots + 1$ is either p_2 or $p_2(\alpha+1)$. Suppose that $p_1^{\alpha} + \cdots + 1 = p_2(\alpha+1)$. Then $(p_2+1)/2$ must be divisible by p_1. But $(p_2+1)/2 = (p_1^{\alpha} + \cdots + p_1 + \alpha + 2)/2(\alpha+1)$. Thus $\alpha+2$ is divisible by p_1 and $\alpha+2 \geqq p_1$. Now, $p_1^{\alpha+1} \equiv p_1 \bmod (\alpha+1)$ and therefore $p_1^{\alpha+1} - 1 \equiv p_1 - 1 \bmod (\alpha+1)$. Since $p_1^{\alpha} + \cdots + 1 \equiv 0 \bmod (\alpha+1)$, $p_1 - 1 \equiv 0 \bmod (\alpha+1)$. Thus $p_1 \geqq \alpha+2$ and consequently $p_1 = \alpha+2$. Since $\alpha+1$ is prime, this is only possible if $\alpha+2 = 3 = p_1$. Thus $(p_2+1)/2 = (3+3)/2(2) = 3/2$, which is impossible for $p_2 \neq 2$.

We then have $p_1^{\alpha} + \cdots + p_1 + 1 = p_2$. If $p_1 \neq 2$,

$$\frac{p_2 + 1}{2} = \frac{p_1^{\alpha} + \cdots + p_1 + 2}{2} \geqq \frac{3^{\alpha} + \cdots + 3 + 2}{2} > \alpha + 1$$

and therefore $(p_2+1)/2$ must be divisible by p_1. This is impossible unless $p_1 = 2$. Thus $p_1 = 2$, $p_2 = 2^{\alpha+1} - 1$, and H is integral, but here n is 2^{α} times the prime $2^{\alpha+1} - 1$ and n is therefore perfect.

If $\alpha+1$ is not prime, let r be its smallest prime factor. Then $\alpha + 1 = rq$, where $q \geqq r$. Here

$$H(n) = \frac{p_1^{\alpha} \cdot p_2 \cdot rq}{\dfrac{p_1 - 1}{p_1 - 1} \cdot \dfrac{p_1^{q} - 1}{p_1^{r} - 1} \cdot \dfrac{p_2 + 1}{2}}.$$

Now, if $q \geqq 4$, $(p_1^{q} - 1)/(p_1 - 1) > rq$ and consequently $(p_1^{q} - 1)/(p_1 - 1)$ must have p_2 as a factor. Also $(p_1^{rq} - 1)/(p_1^{q} - 1) > rq$ and $(p_1^{\alpha+1} - 1)/(p_1 - 1) > p_2 rq = p_2(\alpha+1)$. Thus $H(n)$ can not be integral.

If $q = 3$ and $p_1 \geqq 3$ the same procedure shows that $H(n)$ can not be integral, as is also true of the case $q = 3$, $p_1 = 2$ and $r = 2$. If $q = 3$ and $p_1 = 2$, r can only be 2 or 3, and if r is 3,

$$H(n) = \frac{2^{3} \cdot p_2 \cdot 3^{2}}{7 \cdot 73 \cdot \dfrac{p_2 + 1}{2}}.$$

Thus $H(n)$ can not be integral.

If $q = 2$, r must be 2, and if p_1 is an odd prime, by Theorem 3, $H(n)$ can not be integral. Thus $p_1 = 2$. But then

$$H(n) = \frac{2^3 \cdot p_2 \cdot 4}{3 \cdot 5 \cdot \dfrac{p_2 + 1}{2}}$$

and therefore $H(n)$ can not be integral. This completes the proof.

To obtain the positive integers less than 10^7 with an integral harmonic mean (abbreviated to "H" from now on), it is convenient to construct a table of the values of $H(n)$ for several powers of the smaller primes. Thus $H(2) = 2^2/3$, $H(2^2) = 2^2 \cdot 3/7$, $H(2^3) = 2^5/3 \cdot 5$, $H(3) = 3/2$, $H(3^2) = 3^3/13$, *etc.*

To find the even numbers less than 10^7 with integral H, we start as follows. Let the even number n have the prime factor decomposition $n = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. Since we are interested only in the numbers $n < 10^7$ which have an integral H, by virtue of Theorems 1 and 4, $\alpha_1 \leqq 20$. Now, $H(2^{20}) = 2^{20} \cdot 3/7 \cdot 127 \cdot 337$. Thus if n has integral H and has the factor 2 raised to (exactly) the 20th power, n either contains a prime raised to the 336th power or higher or has 337 as a factor. However, either one of these possibilities makes $n > 10^7$.

When $\alpha_1 = 19$, we have $H(2^{19}) = 2^{21}/3 \cdot 5 \cdot 11 \cdot 31 \cdot 41$. Since $n < 10^7$, n can not contain a prime to the 30th power or higher. This makes $n \geqq 2^{19} \cdot 31 \cdot 41 > 10^7$.

Similarly we dispose of the cases $\alpha_1 = 18, 17, 16$, *etc.*, obtaining no numbers $< 10^7$ with integral H until we arrive at the case $\alpha_1 = 9$. Here $H(2^9) = 2^{10} \cdot 5/3 \cdot 11 \cdot 31$. Thus 31 and 11 must be factors of n. Now since $n < 10^7$, 31^3 can not be a factor of n. If 31^2 were a factor of n,

$$H(2^9 \cdot 31^2) = \frac{2^{10} \cdot 5}{3 \cdot 11 \cdot 31} \cdot \frac{31^2}{331} = \frac{2^{10} \cdot 5 \cdot 31}{3 \cdot 11 \cdot 331}$$

would give 331 as a factor of n and then $n \geqq 2^9 \cdot 11 \cdot 31^2 \cdot 331 > 10^7$. The factor 31 must then appear to the first power in n.

Now,

$$H(2^9 \cdot 31) = \frac{2^{10} \cdot 5}{3 \cdot 11 \cdot 31} \cdot \frac{31}{2^4} = \frac{2^6 \cdot 5}{3 \cdot 11}.$$

Thus 11 must be a factor of n, since otherwise $n \geqq 2^9 \cdot 3^{10} \cdot 31 > 10^7$. If 11^3 were a factor of n, $n \geqq 2^9 \cdot 11^3 \cdot 31 > 10^7$. If n contained 11^2 as a factor,

$$H(2^9 \cdot 31 \cdot 11^2) = \frac{2^6 \cdot 5}{3 \cdot 11} \cdot \frac{11^2 \cdot 3}{7 \cdot 19} = \frac{2^6 \cdot 5 \cdot 11}{7 \cdot 19}$$

would give 19 as a factor of n and then $n \geqq 2^9 \cdot 11^2 \cdot 19 \cdot 31 > 10^7$. The factor 11 must then occur to the first power.

Then

$$H(2^9 \cdot 31 \cdot 11) = \frac{2^6 \cdot 5}{3 \cdot 11} \cdot \frac{11}{2 \cdot 3} = \frac{2^5 \cdot 5}{3^2}.$$

If 3 were not a factor of n, n would have to contain two squares or higher powers or an 8th power at least. Each of these possibilities would make $n>10^7$. Thus 3 must be a factor of n.

If 3^4 were a factor of n, $n\geqq 2^9\cdot 31\cdot 11\cdot 3^4>10^7$. Now,

$$H(2^9\cdot 31\cdot 11\cdot 3^3)=\frac{2^5\cdot 5}{3^2}\cdot\frac{3^3}{2.5}=2^4\cdot 3,$$

which is integral. We thus obtain $n=2^9\cdot 3^3\cdot 11\cdot 31$, a number $<10^7$ with integral H. Observe that we get no more numbers $<10^7$ with integral H by continuing this sequence, since $2^9\cdot 3^3\cdot 11\cdot 31\cdot 5>10^7$. Now if 3^2 were a factor of n, but not 3^3, since

$$H(2^9\cdot 31\cdot 11\cdot 3^2)=\frac{2^5\cdot 5}{3^2}\cdot\frac{3^3}{13}=\frac{2^5\cdot 3\cdot 5}{13},$$

13 would have to be a factor of n and then $n\geqq 2^9\cdot 3^2\cdot 11\cdot 13\cdot 31>10^7$. If 3 were a factor of n but not 3^2, since

$$H(2^9\cdot 31\cdot 11\cdot 3)=\frac{2^5\cdot 5}{3^2}\cdot\frac{3}{2}=\frac{2^4\cdot 5}{3},$$

n would have to contain a square or a higher power. But then $n\geqq 2^9\cdot 31\cdot 11\cdot 3\cdot 5^2>10^7$. We thus conclude the case $\alpha_1=9$.

In a similar manner we consider the cases $\alpha_1=8, 7, 6, 5, 4, 3, 2, 1$, thus obtaining all the even numbers $<10^7$ with integral H and which are tabulated in Table I.

To show that there are no odd numbers $n<10^7$ with integral H, we begin by proving that 3 can not be a factor of n. First, by virtue of Theorem 2, the factor 3 can not appear to an odd power. Thus if 3 is a factor of n, n must be of the form $3^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_r^{\alpha_r}$, where α_1 is even.

If $\alpha_1\geqq 14$, $n\geqq 3^{14}\cdot 5>10^7$. If $\alpha_1=12$, since $H(3^{12})=3^{12}\cdot 13/797161$, 797161 must be a factor of n and then $n>10^7$.

The case $\alpha_1=10$ gives $H(3^{10})=3^{10}\cdot 11/23\cdot 3851$. Therefore $(23)^2$ and $(3851)^2$ would have to be factors of n, by Theorem 2, and then $n>10^7$.

If $\alpha_1=8$, from $H(3^8)=3^{10}/13\cdot 757$ we have that 757 and 13 are both factors of n and $n\geqq 3^8\cdot 13\cdot 757>10^7$.

If $\alpha_1=6$, $H(3^6)=3^6\cdot 7/1093$ shows that 1093 is a factor of n. If $(1093)^2$ were a factor of n, $n\geqq 3^6\cdot(1093)^2>10^7$. Thus 1093 must occur to the first power. Then

$$H(3^6\cdot 1093)=\frac{3^6\cdot 7}{1093}\cdot\frac{1093}{547}=\frac{3^6\cdot 7}{547}.$$

Thus $(547)^2$ is a factor of n and $n>10^7$.

If $\alpha_1=4$, since $H(3^4)=3^4\cdot 5/11^2$, 11 must occur to an even power in n. Thus

$n=3^4\cdot 11^{\alpha_2}\cdot p_3^{\alpha_3}\cdot\cdots p_r^{\alpha_r}$, where α_2 is even. If $\alpha_2\geqq 6$, $n\geqq 3^4\cdot 11^6>10^7$. If $\alpha_2=4$,

$$H(3^4\cdot 11^4)=\frac{3^4\cdot 5}{11^2}\cdot\frac{11^4}{3221}=\frac{3^4\cdot 5\cdot 11^2}{3221}$$

gives that 3221 is a factor of n and thus $n>10^7$. If $\alpha_2=2$,

$$H(3^4\cdot 11^2)=\frac{3^4\cdot 5}{11^2}\cdot\frac{11^2\cdot 3}{7\cdot 19}=\frac{3^5\cdot 5}{7\cdot 19}$$

gives that 7^2 and 19^2 are factors of n and thus $n\geqq 3^4\cdot 11^2\cdot 7^2\cdot 19^2>10^7$. Thus the case $\alpha_1=4$ gives us no numbers $<10^7$ with integral H.

If $\alpha_1=2$, from $H(3^2)=3^2/13$ we have that 13 is a factor of n. If 13^6 is a factor of n, $n\geqq 3^2\cdot 13^6>10^7$. If 13^5 is a factor of n but not 13^6,

$$H(3^2\cdot 13^5)=\frac{3^2}{13}\cdot\frac{13^5}{7\cdot 61\cdot 157}$$

gives that 157 is a factor of n and $n>10^7$. If 13^4 is a factor of n but not 13^5,

$$H(3^2\cdot 13^4)=\frac{3^2}{13}\cdot\frac{13^4\cdot 5}{30941}$$

gives that 30941 is a factor of n and $n>10^7$. If 13^3 is a factor of n but not 13^4,

$$H(3^2\cdot 13^3)=\frac{3^2}{13}\cdot\frac{13^3}{5\cdot 7\cdot 17}=\frac{3^2\cdot 13^2}{5\cdot 7\cdot 17}$$

gives that 7 must be a factor of n, since otherwise $n\geqq 3^2\cdot 13^3\cdot 5^6\cdot>10^7$. By Theorem 2, 7^2 is also a factor of n and we also have that 17 and 5 are factors of n. Thus $n\geqq 3^2\cdot 13^3\cdot 5\cdot 7^2\cdot 17>10^7$. If 13^2 is a factor of n but not 13^3,

$$H(3^2\cdot 13^2)=\frac{3^2}{13}\cdot\frac{13^2}{61}=\frac{3^2\cdot 13}{61}$$

gives that 61 is a factor of n and since $n<10^7$, 61 can only occur to the first or second power. If 61 occurs to the second power,

$$H(3^2\cdot 13^2\cdot 61^2)=\frac{3^2\cdot 13}{61}\cdot\frac{61^2}{13\cdot 97}=\frac{3^2\cdot 61}{97}$$

gives that 97 is a factor of n and $n\geqq 3^2\cdot 13^2\cdot 61^2\cdot 97>10^7$. Thus 61 occurs to the first power only. Then

$$H(3^2\cdot 13^2\cdot 61)=\frac{3^2\cdot 13}{61}\cdot\frac{61}{31}=\frac{3^2\cdot 13}{31}$$

gives that 31^2 is a factor of n and $n\geqq 3^2\cdot 13^2\cdot 31^2\cdot 61>10^7$. We have thus shown that 3 can not be a factor of any number $<10^7$ with integral H.

In a similar manner it may be shown that 5 can not be a factor of n, although in this case odd powers of 5 must also be considered. The work is continued showing that 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, and 53 can not be factors of n. This is all that is necessary, for by virtue of Theorems 1, 3 and 4, if n does not have any prime <59 as a factor and has integral H, $n \geqq 59^2 \cdot 61^2 > 10^7$.

Following is a table of the forty-five positive integers less than 10^7 which have integral H:

TABLE I

n	$H(n)$	n	$H(n)$
$6=2\cdot 3$	2	$242060=2^2\cdot 5\cdot 7^2\cdot 13\cdot 19$	26
$28=2^2\cdot 7$	3	$332640=2^5\cdot 3^3\cdot 5\cdot 7\cdot 11$	44
$140=2^2\cdot 5\cdot 7$	5	$360360=2^3\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 13$	44
$270=2\cdot 3^3\cdot 5$	6	$539400=2^3\cdot 3\cdot 5^2\cdot 29\cdot 31$	29
$496=2^4\cdot 31$	5	$695520=2^5\cdot 3^3\cdot 5\cdot 7\cdot 23$	46
$672=2^5\cdot 3\cdot 7$	8	$726180=2^2\cdot 3\cdot 5\cdot 7^2\cdot 13\cdot 19$	39
$1638=2\cdot 3^2\cdot 7\cdot 13$	9	$753480=2^3\cdot 3^2\cdot 5\cdot 7\cdot 13\cdot 23$	46
$2970=2\cdot 3^3\cdot 5\cdot 11$	11	$950976=2^6\cdot 3^3\cdot 13\cdot 127$	27
$6200=2^3\cdot 5^2\cdot 31$	10	$1089270=2\cdot 3^2\cdot 5\cdot 7^2\cdot 13\cdot 19$	42
$8128=2^6\cdot 127$	7	$1421280=2^5\cdot 3^2\cdot 5\cdot 7\cdot 47$	47
$8190=2\cdot 3^2\cdot 5\cdot 7\cdot 13$	15	$1539720=2^3\cdot 3^3\cdot 5\cdot 7\cdot 13\cdot 47$	47
$18600=2^3\cdot 3\cdot 5^2\cdot 31$	15	$2178540=2^2\cdot 3^3\cdot 5\cdot 7^2\cdot 13\cdot 19$	54
$18620=2^2\cdot 5\cdot 7^2\cdot 19$	14	$2229500=2^2\cdot 5^3\cdot 7^2\cdot 13$	35
$27846=2\cdot 3^3\cdot 7\cdot 13\cdot 17$	17	$2290260=2^2\cdot 3\cdot 5\cdot 7^2\cdot 19\cdot 41$	41
$30240=2^5\cdot 3^3\cdot 5\cdot 7$	24	$2457000=2^3\cdot 3^3\cdot 5^3\cdot 7\cdot 13$	60
$32760=2^3\cdot 3^3\cdot 5\cdot 7\cdot 13$	24	$2845800=2^3\cdot 3^4\cdot 5^2\cdot 17\cdot 31$	51
$55860=2^2\cdot 3\cdot 5\cdot 7^2\cdot 19$	21	$4358600=2^3\cdot 5^2\cdot 19\cdot 31\cdot 37$	37
$105664=2^6\cdot 13\cdot 127$	13	$4713984=2^9\cdot 3^2\cdot 11\cdot 31$	48
$117800=2^3\cdot 5^2\cdot 19\cdot 31$	19	$4754880=2^6\cdot 3^2\cdot 5\cdot 13\cdot 127$	45
$167400=2^3\cdot 3^3\cdot 5^2\cdot 31$	27	$5772200=2^3\cdot 5^2\cdot 7^3\cdot 19\cdot 31$	49
$173600=2^5\cdot 5^2\cdot 7\cdot 31$	25	$6051500=2^2\cdot 5^3\cdot 7^3\cdot 13\cdot 19$	50
$237510=2\cdot 3^4\cdot 5\cdot 7\cdot 13\cdot 29$	29	$8506400=2^5\cdot 5^2\cdot 7^3\cdot 31$	49
		$8872200=2^3\cdot 3^3\cdot 5^2\cdot 31\cdot 53$	53

From a number n with integral H, another number np having integral H may sometimes be obtained when p is an odd prime that does not divide n. A new such number np will be obtained when $H(n)$ is divisible by $\frac{1}{2}(p+1)$. For example, if we let $n=4713984$, we find in the manner just outlined that np also has integral H for the values of $p=5$, 7, 23 and 47.

We can also find some numbers of the form np^2 with integral H when n has integral H and is relatively prime to p. For example, from $n=2^3\cdot 5^2\cdot 7^2\cdot 13\cdot 19\cdot 31$ and $n=2^5\cdot 5^2\cdot 7^2\cdot 13\cdot 31$, both of which have $H(n)=91$, we obtain respectively $2^3\cdot 3^2\cdot 5^2\cdot 7^2\cdot 13\cdot 19\cdot 31$ and $2^5\cdot 3^2\cdot 5^2\cdot 7^2\cdot 13\cdot 31$, whose harmonic mean is 189. In a similar manner, from $n=2^2\cdot 5^3\cdot 7^2\cdot 13\cdot 19$ ($H=50$) we obtain $2^2\cdot 3^3\cdot 5^3\cdot 7^2\cdot 13\cdot 19$ ($H=135$) by multiplying n by 3^3.

From Table I one may be led to the conjecture that every positive integer with integral H must necessarily have its largest factor in its prime factor decomposition appear to the first power. However, several numbers were found which show that this conjecture is false, among them

$$n_1 = 2^8 \cdot 3^3 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 41 \cdot 71 \cdot 73^3 \qquad (H = 2 \cdot 3 \cdot 73^2)$$

$$n_2 = 2^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13^2 \cdot 19 \cdot 31^3 \cdot 37^3 \cdot 61 \cdot 73 \cdot 137^2 \qquad (H = 2^2 \cdot 5 \cdot 13 \cdot 137)$$

and

$$n_3 = 2^9 \cdot 3^2 \cdot 5 \cdot 7^3 \cdot 11 \cdot 13^2 \cdot 31^3 \cdot 37^2 \cdot 61 \cdot 67^2 \qquad (H = 2^3 \cdot 37 \cdot 67).$$

It is not difficult to obtain numbers having integral H by means of a table of values of H for powers of primes. Over two hundred numbers with integral H and larger than 10^7 were found, among them the following three:

$$2^{126} \cdot 19^2 \cdot (2^{127} - 1) \qquad (H = 19^2),$$

$$2^{47} \cdot 3^{15} \cdot 5^8 \cdot 7^3 \cdot 11^3 \cdot 13^4 \cdot 17^3 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31^2 \cdot 41 \cdot 43 \cdot 53 \cdot 61 \cdot 83^2 \cdot 97^2 \cdot 127 \cdot 191 \cdot 193$$
$$\cdot 241 \cdot 257 \cdot 317 \cdot 331 \cdot 337 \cdot 367 \cdot 673 \cdot 829 \cdot 3169 \cdot 30941 \qquad (H = 2^{29} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17),$$

$$2^{53} \cdot 3^{19} \cdot 5^{15} \cdot 7^{11} \cdot 11^7 \cdot 13^3 \cdot 17^3 \cdot 19^5 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43 \cdot 61^3 \cdot 73 \cdot 79^3 \cdot 127 \cdot 131 \cdot 157 \cdot 181$$
$$\cdot 197 \cdot 199 \cdot 223 \cdot 313 \cdot 383 \cdot 523 \cdot 1181 \cdot 1861 \cdot 3121 \cdot 7321 \cdot 11489 \cdot 11939 \cdot 21803$$
$$\cdot 87211 \cdot 262657 \qquad (H = 2^{16} \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 17 \cdot 19 \cdot 61 \cdot 79).$$

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GEOMETRICAL PROPERTIES OF TWO-DIMENSIONAL WAVE MOTION*

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1. Introduction. We consider solutions of the differential equation

$$u_{xx} + u_{yy} = c^{-2} u_{tt} \tag{1}$$

of the form

$$u = \exp\left[\alpha(x, y) + i\beta(x, y) + i\omega t\right]. \tag{2}$$

Here x and y denote rectangular coordinates in a plane, t denotes the time, and c and ω are positive constants. The subscripts indicate partial differentiation as

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usual. $\alpha(x, y)$ and $\beta(x, y)$ are real-valued functions, which we assume to be of class C^4 in an open connected domain R to which the considerations are restricted.

Such a solution will be called a sinusoidal wave. The importance of these waves in physics is well known, and requires no discussion.

The function u defined by (2) is a solution of (1) if, and only if, α and β satisfy the system of equations

$$\begin{aligned} &\alpha_{xx} + \alpha_{yy} + \alpha_x^2 + \alpha_y^2 - \beta_x^2 - \beta_y^2 + k^2 = 0, \qquad k^2 = \omega^2 c^{-2}, \\ &\beta_{xx} + \beta_{yy} + 2(\alpha_x\beta_x + \alpha_y\beta_y) = 0. \end{aligned} \tag{3}$$

It is clear that α and β cannot both be constants. There are cases, which will be discussed in the subsequent sections, in which one of the functions is a constant. Ordinarily, however, neither function is a constant, and for the moment we consider only this ordinary case.

The level curves of the function α in the domain R will be called curves of constant amplitude, or briefly α-curves. Similarly, the level curves of β in R will be called curves of constant phase, or briefly β-curves.

The chief problem considered in the paper can now be stated in the following form: To determine a set of properties which are necessary and sufficient in order that two given families of curves in the domain R shall be, respectively, the family of α-curves and the family of β-curves for some sinusoidal wave.

2. Waves of constant amplitude. If α is a constant, we have what we call a sinusoidal wave of constant amplitude. This degenerate case gives rise to the following problem: To determine a set of properties which are necessary and sufficient in order that a given family of curves in the domain R shall be the family of β-curves for some sinusoidal wave of constant amplitude.

The simple solution of this problem is easily obtained by means of the methods developed in the subsequent sections. We shall merely state the result in the form of the following theorem.

THEOREM 1. *In order that a family of curves shall be the family of β-curves for some sinusoidal wave of constant amplitude, it is necessary and sufficient that it be a family of parallel straight lines.*

3. Standing waves. If β is a constant, we have a standing sinusoidal wave. In this degenerate case the system of equations (3) reduces to the single equation,

$$\alpha_{xx} + \alpha_{yy} + \alpha_x^2 + \alpha_y^2 + k^2 = 0, \tag{4}$$

and we have the following problem: To determine a set of properties which are necessary and sufficient in order that a given family of curves in the domain R shall be the family of α-curves for some sinusoidal standing wave.

Before proceeding to the solution of this problem, we shall give some formulae which will be used throughout the rest of the paper.

Consider a family of curves which is to be specialized so that it becomes the family of α-curves for some standing wave. Anticipating the final result, we call the curves α-curves even before the necessary specialization has been imposed. It is understood that through each point of the domain R there passes just one α-curve.

Let the α-curves be oriented so that the positive senses on neighboring curves are concordant. Let $T(x, y)$ denote the positively directed tangent to the typical α-curve at the typical point (x, y). Let $\theta(x, y)$ denote the angle which $T(x, y)$ makes with the positive x-axis. For the sake of the later developments, we assume that $\theta(x, y)$ is of class C^2 in R. The function $\theta(x, y)$ characterizes the family of α-curves, in the sense that the family is defined by the differential equation

$$\frac{dy}{dx} = \tan \theta(x, y).$$

Similarly, any congruence of smooth curves is characterized by a point function having a geometrical significance which is analogous to that of $\theta(x, y)$. In particular, the family of (suitably oriented) orthogonal trajectories of the α-curves is characterized by the function $\theta^*(x, y) = \theta(x, y) + \pi/2$.

Let F be any differentiable point function. The rate of change of F with respect to arc length along an α-curve is

$$\frac{dF}{ds} = F_x \cos \theta + F_y \sin \theta, \tag{5}$$

and the rate of change of F with respect to arc length along an orthogonal trajectory of the α-curves is

$$\frac{dF}{ds^*} = F_x \cos \theta^* + F_y \sin \theta^* = -F_x \sin \theta + F_y \cos \theta. \tag{5'}$$

By (5) and (5'), we have the relations

$$\begin{aligned} F_x &= \frac{dF}{ds} \cos \theta - \frac{dF}{ds^*} \sin \theta, \\ F_y &= \frac{dF}{ds} \sin \theta + \frac{dF}{ds^*} \cos \theta. \end{aligned} \tag{6}$$

In particular, the curvature of the typical α-curve at the typical point is

$$\gamma = \frac{d\theta}{ds} = \theta_x \cos \theta + \theta_y \sin \theta, \tag{7}$$

and the curvature of the typical orthogonal trajectory of the α-curves, at the typical point, is

$$\gamma^* = \frac{d\theta^*}{ds^*} = -\theta_x \sin\theta + \theta_y \cos\theta. \tag{7'}$$

By (7) and (7'), we have the relations

$$\begin{aligned} \theta_x &= \gamma \cos\theta - \gamma^* \sin\theta, \\ \theta_y &= \gamma \sin\theta + \gamma^* \cos\theta. \end{aligned} \tag{8}$$

We now proceed to the solution of the problem stated at the beginning of this section.

If the curves under consideration are the level curves of a function $\alpha(x, y)$, there exists a pair of relations of the form

$$\alpha_x = \lambda \sin\theta, \qquad \alpha_y = -\lambda \cos\theta, \tag{9}$$

where λ is a point function.

The function λ is subject to two conditions. First, we have the integrability condition for the system (9), namely,

$$(\lambda \sin\theta)_y = (-\lambda \cos\theta)_x.$$

In the second place, if α is to satisfy (4), we must have

$$(\lambda \sin\theta)_x + (-\lambda \cos\theta)_y + \lambda^2 + k^2 = 0.$$

Regarded as a system of equations, these two equations are equivalent to the system

$$\begin{aligned} \lambda_x &= -k^2 \sin\theta - \lambda\theta_y - \lambda^2 \sin\theta, \\ \lambda_y &= k^2 \cos\theta + \lambda\theta_x + \lambda^2 \cos\theta. \end{aligned} \tag{10}$$

Using (5), (5'), (7), and (7'), we find that (10) is equivalent to the system

$$\begin{aligned} \frac{d\lambda}{ds} &= -\gamma^*\lambda, \\ \frac{d\lambda}{ds^*} &= k^2 + \gamma\lambda + \lambda^2. \end{aligned} \tag{11}$$

The integrability condition for the system (10), regarded as a system in the unknown λ, can be written in the form

$$(\theta_{xx} + \theta_{yy})\lambda + 2k^2(-\theta_x \sin\theta + \theta_y \cos\theta) = 0, \tag{12}$$

or in the following intrinsic form

$$\left[\frac{d\gamma}{ds} + \frac{d\gamma^*}{ds^*}\right]\lambda + 2k^2\gamma^* = 0. \tag{13}$$

The equations (10) and (12), or equivalently (11) and (13), are the fundamental relations upon which the solution of the problem depends.

We can satisfy (13) by setting

$$\gamma^* \equiv 0, \qquad \frac{d\gamma}{ds} \equiv 0. \tag{14}$$

This pair of identities implies that the α-curves are parallel straight lines or concentric circles. Standing waves of each of these types do exist. In fact, when (12) is satisfied in virtue of (14) the system (10) is complete. The system possesses a unique solution which assumes an arbitrarily prescribed value at an arbitrarily prescribed point. When such a solution has been obtained, the equations (9) determine a standing wave with an amplitude which is constant along each curve of the family.

Since α is not to be a constant in the present case, we cannot have $\lambda \equiv 0$. Hence, by (13), it is impossible to have just one of the quantities

$$\frac{d\gamma}{ds} + \frac{d\gamma^*}{ds^*}, \qquad \gamma^*$$

identically zero. Having disposed of the case in which both of the quantities are identically zero, we now turn to the case in which neither is zero.

In this case we can satisfy (13) by setting

$$\lambda = -2k^2\gamma^* A^{-1}, \tag{15}$$

where

$$A = \frac{d\gamma}{ds} + \frac{d\gamma^*}{ds^*}.$$

In general, the λ determined by (15), for a particular family of curves, will not satisfy the equations (11). However, in the present case the family is still unspecialized, and the problem is precisely that of specializing the family so that the λ given by (15) will satisfy (11). Thus, in order that a family of curves, which is neither a family of parallel straight lines nor a family of concentric circles, shall be the family of α-curves for some standing wave, it is necessary that the family possess the properties expressed by the following equations:

$$\begin{aligned} \frac{d}{ds}(A/\gamma^*) &= A, \\ \frac{d}{ds^*}(A/\gamma^*) &= 2k^2 - A\gamma/\gamma^* + A^2/(2\gamma^{*2}). \end{aligned} \tag{16}$$

Conversely, the set of properties expressed by these equations is easily seen to be also sufficient.

The following theorem summarizes the results which have been obtained.

THEOREM 2. *In order that a family of curves shall be the family of α-curves for some sinusoidal standing wave, it is necessary and sufficient that it be a family of parallel straight lines, a family of concentric circles, or a family possessing the properties expressed by equations* (16).

4. Progressive waves with non-constant amplitude. Proceeding now to the case in which neither α nor β is a constant, we consider two families of curves in the domain R, the one characterized by an angle $\theta(x, y)$, the other characterized by an angle $\phi(x, y)$. It is required to specialize the angles so that the first and second families of curves shall be, respectively, the family of α-curves and the family of β-curves for some progressive sinusoidal wave with non-constant amplitude.

If the curves of the first family are the level curves of a function $\alpha(x, y)$, and the curves of the second family are the level curves of a function $\beta(x, y)$, we have the relations

$$\alpha_x = \lambda \sin\theta \qquad \alpha_y = -\lambda\cos\theta, \tag{17}$$

$$\beta_x = \mu \sin\phi, \qquad \beta_y = -\mu\cos\phi, \tag{18}$$

where λ and μ are point functions.

We must have the integrability conditions

$$(\lambda \sin\theta)_y = (-\lambda\cos\theta)_x, \tag{19}$$

$$(\mu \sin\phi)_y = (-\mu\cos\phi)_x. \tag{20}$$

Also, if the functions α and β are to satisfy equations (3), we must have the relations

$$(\lambda \sin\theta)_x + (-\lambda\cos\theta)_y + \lambda^2 - \mu^2 + k^2 = 0, \tag{21}$$

$$(\mu \sin\phi)_x + (-\mu\cos\phi)_y + 2\lambda\mu\cos(\theta - \phi) = 0. \tag{22}$$

The system composed of equations (19), $\cdots$, (22) is equivalent to the following:

$$\begin{aligned}
\lambda_x &= -k^2\sin\theta - \lambda\theta_y - (\lambda^2 - \mu^2)\sin\theta,\\
\lambda_y &= k^2\cos\theta + \lambda\theta_x + (\lambda^2 - \mu^2)\cos\theta,\\
\mu_x &= -\mu\phi_y - 2\lambda\mu\cos(\theta - \phi)\sin\phi,\\
\mu_y &= \mu\phi_x + 2\lambda\mu\cos(\theta - \phi)\cos\phi.
\end{aligned} \tag{23}$$

The integrability conditions for (23), regarded as a system of partial differential equations in the unknowns λ and μ, can be written in the form

$$\begin{aligned}
\lambda\mu^2\sin 2(\theta - \phi) - [(\theta - \phi)_x\sin\theta - (\theta - \phi)_y\cos\theta]\mu^2\\
- \tfrac{1}{2}(\theta_{xx} + \theta_{yy})\lambda + k^2(\theta_x\sin\theta - \theta_y\cos\theta) = 0,
\end{aligned} \tag{24}$$

$$(25)\qquad \begin{aligned}(\lambda^2-\mu^2)\sin 2(\theta-\phi)+2[(\theta-\phi)_x\sin(\theta-2\phi)\\ +(\theta-\phi)_y\cos(\theta-2\phi)]\lambda-[\phi_{xx}+\phi_{yy}-k^2\sin 2(\theta-\phi)]=0.\end{aligned}$$

(We have removed a factor μ from the left-hand member of the second condition. This is permissible, since μ cannot be zero in the case under consideration.)

Before proceeding further, we shall put the fundamental equations (23), (24), (25) into their simpler intrinsic forms. We shall write $\theta-\phi=\delta$, so that δ denotes the angle between the α-curve and the β-curve passing through the typical point.

It is easily seen that the system of equations (23) is equivalent to the following:

$$(26)\qquad \begin{aligned}\frac{d\lambda}{ds_\alpha}&=-\lambda\gamma_\alpha^*, & \frac{d\lambda}{ds_\alpha^*}&=k^2+\lambda\gamma_\alpha+\lambda^2-\mu^2,\\ \frac{d\mu}{ds_\beta}&=-\mu\gamma_\beta^*, & \frac{d\mu}{ds_\beta^*}&=\mu\gamma_\beta+2\lambda\mu\cos\delta.\end{aligned}$$

The symbols s_α, s_α^*, s_β, and s_β^*, respectively, denote the arc lengths along the α-curve, the orthogonal trajectory of the α-curves, the β-curve, and the orthogonal trajectory of the β-curves passing through the typical point. The symbols γ_α, γ_α^*, γ_β, and γ_β^* denote the curvatures of the several curves in an obvious way.

The integrability conditions (24) and (25) can be written in intrinsic form as follows:

$$(27)\qquad \lambda\mu^2\sin 2\delta+\frac{d\delta}{ds_\alpha^*}\mu^2-\frac{1}{2}\left[\frac{d\gamma_\alpha}{ds_\alpha}+\frac{d\gamma_\alpha^*}{ds_\alpha^*}\right]\lambda-k^2\gamma_\alpha^*=0,$$

$$(28)\qquad \begin{aligned}(\lambda^2-\mu^2)\sin 2\delta+2\left[\frac{d\delta}{ds_\beta}\sin\delta+\frac{d\delta}{ds_\beta^*}\cos\delta\right]\lambda\\ -\left[\frac{d\gamma_\beta}{ds_\beta}+\frac{d\gamma_\beta^*}{ds_\beta^*}-k^2\sin 2\delta\right]=0.\end{aligned}$$

In considering the consequences of the above equations, let us first assume that $\sin 2\delta$ is not identically zero, and let us restrict our attention to a domain R in which $\sin 2\delta$ is nowhere zero.

The solution of the problem is obtained from equations (26), (27), (28) in a way which is entirely analogous to that in which the solution of the problem of the preceding section was obtained from equations (11), (13). The reasoning is straightforward, and it will suffice to state the result in the form of the following theorem.

THEOREM 3. *In order that two distinct non-orthogonal families of curves in a domain R shall be, respectively, the family of α-curves and the family of β-curves for some progressive sinusoidal wave with non-constant amplitude, it is necessary*

and sufficient that the system of equations (27), (28) *have a real solution* (λ, μ) *satisfying the equations* (26).

The solution of equations (27), (28) for λ and μ involves only rational operations, the extraction of a square root, and the solution of a cubic equation. In principle, therefore, the necessary and sufficient condition, which is stated in an implicit form in the theorem, can be stated in a fully explicit form in terms of the geometrical quantities δ, γ_α, γ_β, $d\gamma_\alpha/ds_\alpha$, *etc.* However, because of the complexity of the explicit condition, it seems not worth while to formulate it here.

5. Waves for which $\sin 2\delta \equiv 0$. We now consider the case in which $\sin 2\delta \equiv 0$, noting that this implies $\delta \equiv 0$, $\delta \equiv \pm\pi/2$, or $\delta \equiv \pi$.

In this case the equations (27), (28) reduce to the forms

$$\left[\frac{d\gamma_\alpha}{ds_\alpha} + \frac{d\gamma_\alpha^*}{ds_\alpha^*}\right]\lambda + 2k^2\gamma_\alpha^* = 0, \qquad \frac{d\gamma_\beta}{ds_\beta} + \frac{d\gamma_\beta^*}{ds_\beta^*} = 0. \tag{29}$$

The geometrical meaning of the condition $\sin 2\delta \equiv 0$ implies the relation

$$\frac{d\gamma_\alpha}{ds_\alpha} + \frac{d\gamma_\alpha^*}{ds_\alpha^*} \equiv \frac{d\gamma_\beta}{ds_\beta} + \frac{d\gamma_\beta^*}{ds_\beta^*}. \tag{30}$$

By (29) and (30), we have the equations

$$\frac{d\gamma_\alpha}{ds_\alpha} + \frac{d\gamma_\alpha^*}{ds_\alpha^*} = 0, \qquad \gamma_\alpha^* = 0. \tag{31}$$

The equations (31) imply that the α-curves can only be parallel straight lines or concentric circles. We know that there do exist progressive waves for which the α-curves are of these types, and for which the β-curves coincide with the α-curves or are the orthogonal trajectories of the α-curves. Hence, we can state the following theorem.

THEOREM 4. *In order that a family of curves shall be the family of α-curves for some progressive sinusoidal wave with a non-constant amplitude, and with the β-curves either coinciding with the α-curves or being the orthogonal trajectories of the α-curves, it is necessary and sufficient that the family be a family of parallel straight lines or a family of concentric circles.*

MATHEMATICAL NOTES

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TWO SERIES OF PARTITIONS

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Denoting the number of partitions of n by $p(n)$, so that

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots} = \sum_0^\infty p(n)q^n,$$

we prove

$$\sum_0^\infty \frac{p(n)}{\sqrt{n-\frac{1}{24}+\theta}} \frac{\sinh 2\pi\sqrt{\frac{2}{3}(n-\frac{1}{24}+\theta)}}{\cosh 3\pi\sqrt{\frac{2}{3}(n-\frac{1}{24}+\theta)}} = \frac{1}{\pi\theta\sqrt{2}}$$

and

$$\sum_0^\infty \frac{p(n)(n-\theta)}{\cosh \pi\sqrt{2n-\frac{1}{4}+4\theta}} = 0, \qquad (\theta > 0).$$

The proofs are based on termwise integrations of Euler's and Jacobi's expansions

$$q^{1/24}(1-q)(1-q^2)(1-q^3)\cdots = q^{1^2/24} - q^{5^2/24} - q^{7^2/24} + q^{11^2/24} + \cdots,$$

$$q^{1/8}((1-q)(1-q^2)(1-q^3)\cdots)^3 = 1q^{1^2/8} - 3q^{3^2/8} + 5q^{5^2/8} - 7q^{7^2/8} + \cdots,$$

which are shown to lead to the definite integrals

$$\int_0^1 q^{\alpha+1/24}(1-q)(1-q^2)(1-q^3)\cdots\frac{dq}{q} = \frac{\pi\sqrt{2}}{\sqrt{\alpha}}\frac{\sinh \frac{1}{3}\pi\sqrt{24\alpha}}{\cosh \frac{1}{2}\pi\sqrt{24\alpha}}, \qquad (\alpha > -\tfrac{1}{24}),$$

$$\int_0^1 q^{\beta+1/8}((1-q)(1-q^2)(1-q^3)\cdots)^3\frac{dq}{q} = \frac{2\pi}{\cosh \pi\sqrt{2\beta}}, \qquad (\beta > -\tfrac{1}{8}).$$

We note that Euler's expansion can be written

$$q^{1/24}(1-q)(1-q^2)(1-q^3)\cdots = \sum_1^\infty q^{n^2/24}\frac{\omega^n - \omega^{5n} - \omega^{7n} + \omega^{11n}}{\sqrt{12}},$$

where

$$\omega = e^{2\pi i/12}.$$

Termwise integration yields

$$\int_0^1 q^{\alpha+1/24}(1-q)(1-q^2)(1-q^3)\cdots\frac{dq}{q}$$

$$=\sum_1^\infty \frac{1}{\alpha+n^2/24}\,\frac{\omega^n-\omega^{5n}-\omega^{7n}+\omega^{11n}}{\sqrt{12}},\quad (\alpha>-\tfrac{1}{24}).$$

Now

$$\frac{1}{\pi}\sum_1^\infty\left\{\frac{z\cos 2n\pi a}{z^2+n^2}-\frac{n\sin 2n\pi a}{z^2+n^2}\right\}=\frac{e^{2\pi az}}{e^{2\pi z}-1}-\frac{1}{2\pi z},\quad (0<a<1).$$

Therefore

$$\frac{1}{\pi}\sum_1^\infty\frac{z}{z^2+n^2}(\omega^n-\omega^{5n}-\omega^{7n}+\omega^{11n})=\frac{e^{2\pi z/12}-e^{10\pi z/12}-e^{14\pi z/12}+e^{22\pi z/12}}{e^{2\pi z}-1}$$

$$=\frac{\sinh\frac{1}{2}\pi z}{\cosh\frac{1}{2}\pi z}.$$

Hence

$$\int_0^1 q^{\alpha+1/24}(1-q)(1-q^2)(1-q^3)\cdots\frac{dq}{q}=\frac{\pi\sqrt{2}}{\sqrt{\alpha}}\,\frac{\sinh\frac{1}{2}\pi\sqrt{24\alpha}}{\cosh\frac{1}{2}\pi\sqrt{24\alpha}},\quad (\alpha>-\tfrac{1}{24}),$$

our first definite integral.

We write this as

$$\int_0^1 q^{\theta+n}(1-q)(1-q^2)(1-q^3)\cdots\frac{dq}{q}$$

$$=\frac{\pi\sqrt{2}}{\sqrt{n-\frac{1}{24}+\theta}}\,\frac{\sinh 2\pi\sqrt{\tfrac{2}{3}(n-\frac{1}{24}+\theta)}}{\cosh 3\pi\sqrt{\tfrac{2}{3}(n-\frac{1}{24}+\theta)}},\qquad(\theta>0),$$

where n is a non-negative integer. Multiplying across by $p(n)$ and summing with respect to n from 0 to ∞, we have

$$\pi\sqrt{2}\sum_0^\infty\frac{p(n)}{\sqrt{n-\frac{1}{24}+\theta}}\,\frac{\sinh 2\pi\sqrt{\tfrac{2}{3}(n-\frac{1}{24}+\theta)}}{\cosh 3\pi\sqrt{\tfrac{2}{3}(n-\frac{1}{24}+\theta)}}$$

$$=\int_0^1\left(\sum_0^\infty p(n)q^n\right)q^\theta(1-q)(1-q^2)(1-q^3)\cdots\frac{dq}{q}$$

$$=\int_0^1\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}\,q^\theta(1-q)(1-q^2)(1-q^3)\cdots\frac{dq}{q}$$

$$=\int_0^1 q^\theta\frac{dq}{q}=\frac{1}{\theta},\qquad(\theta>0),$$

giving our first formula.

Similarly, from Jacobi's identity, we have

$$\int_0^1 q^{\beta+1/8}((1-q)(1-q^2)(1-q^3)\cdots)^3\frac{dq}{q}$$
$$=\frac{1}{\beta+1^2/8}-\frac{3}{\beta+3^2/8}+\frac{5}{\beta+5^2/8}-\cdots$$
$$=\frac{2\pi}{\cosh\pi\sqrt{2\beta}},\qquad(\beta>-\tfrac{1}{8}),$$

which we write as

$$\int_0^1 q^{3\theta}((1-q)(1-q^2)(1-q^3)\cdots)^3dq^{n-\theta}=\frac{2\pi(n-\theta)}{\cosh\pi\sqrt{2n-\frac{1}{4}+4\theta}},\qquad(\theta>0).$$

Multiplying across by $p(n)$ and summing with respect to n from 0 to ∞, we have

$$2\pi\sum_0^\infty\frac{p(n)(n-\theta)}{\cosh\pi\sqrt{2n-\frac{1}{4}+4\theta}}$$
$$=\int_0^1 q^{3\theta}((1-q)(1-q^2)(1-q^3)\cdots)^3d\,\frac{1}{q^\theta(1-q)(1-q^2)(1-q^3)\cdots}$$
$$=-\tfrac{1}{2}\left[q^{2\theta}((1-q)(1-q^2)(1-q^3)\cdots)^2\right]_0^1=0,\qquad(\theta>0),$$

our second result.

NOTE ON THE CYCLOTOMIC POLYNOMIAL

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1. Let p denote a prime $\geqq 3$. It is well-known that

$$4\frac{x^p-1}{x-1}=Y^2(x)-(-1)^{\frac{1}{2}(p-1)}pZ^2(x),\tag{1.1}$$

where $Y(x)$ and $Z(x)$ are polynomials with integral coefficients, deg $Y(x)=\frac{1}{2}(p-1)$, deg $Z(x)<\frac{1}{2}(p-1)$. For references see [1; §16], [2; Chapter 7]. If we put $x=t+1$, (1) becomes

$$4\frac{(t+1)^p-1}{t}=U^2(t)-\epsilon pV^2(t),\qquad(\epsilon=(-1)^{\frac{1}{2}(p-1)}),\tag{1.2}$$

where $U(t)=Y(t+1)$, $V(t)=Z(t+1)$. Let

$$V(t)=b_0+b_1t+\cdots+b_{m-1}t^{m-1},\qquad(m=\tfrac{1}{2}(p-1)),$$

then we show that the b_r are determined (mod p) by means of

$$(1.3) \qquad 2\eta\left(\sum_0^{m-1}\frac{(-1)^r}{r+1}t^r\right)^{1/2} \equiv V(t) + \text{terms of higher degree (mod } p),$$

where η is a number such that $\eta^2 \equiv -\epsilon \pmod{p}$. If we put

$$U(t) = a_0 + a_1 t + \cdots + a_{m-1}t^{m-1} + 2t^m,$$

then we have $p \mid a_r$ and

$$(1.4) \qquad \frac{1}{p}a_r \equiv \frac{(-1)^{r+m}}{r+m+1} + \epsilon b'_{r+m} \pmod{p}, \qquad (0 \leqq r \leqq m-1),$$

where $V^2(t) = \sum b'_r t^r$.

2. In the first place we have

$$\frac{(t+1)^p - 1}{t} = \sum_{r=1}^{p-1}\binom{p}{r}t^{r-1} + t^{p-1}.$$

Since

$$\binom{p}{r} = \frac{p}{r}\binom{p-1}{r-1} \equiv (-1)^{r-1}\frac{p}{r} \pmod{p^2}, \qquad (1 \leqq r \leqq p-1),$$

we see that

$$(2.1) \qquad \frac{(t+1)^p - 1}{t} \equiv p\sum_{r=1}^{p-1}\frac{(-1)^{r-1}}{r}t^{r-1} + t^{p-1} \pmod{p^2}.$$

It is evident from (1.2) and (2.1) that $U(t) \equiv 2t^m \pmod{p}$, so that $a_r \equiv 0 \pmod{p}$. We write $a_r = pc_r$. Thus (1.2) becomes

$$(2.2) \qquad \begin{aligned} &4p\sum_{r=1}^{p-1}\frac{(-1)^{r-1}}{r}t^{r-1} + t^{p-1} \\ &\qquad \equiv \left(p\sum_0^{m-1}c_r t^r + 2t^m\right)^2 - \epsilon p\left(\sum_0^{m-1}b_r t^r\right) \pmod{p^2}. \end{aligned}$$

We examine the coefficients of x^r, $0 \leqq r \leqq m-1$, in both members of (2.2). Clearly the term $U^2(t)$ contributes nothing; consequently (2.2) implies that the congruence

$$(2.3) \qquad 4\sum_{r=0}^{m-1}\frac{(-1)^r}{r+1}t^r \equiv -\epsilon\left(\sum_0^{m-1}b_r t^r\right)^2 \pmod{p}$$

is valid for terms of degree $\leqq m-1$. This is evidently equivalent to (1.3).

Next (2.2) yields

$$4\sum_{r=0}^{p-2}\frac{(-1)^r}{r+1}t^r+\epsilon V^2(t)\equiv 4\sum_{0}^{m-1}c_r t^{r+m}\pmod{p}, \tag{2.4}$$

where now we ignore terms of degree $<m$. It is clear that (2.4) is equivalent to (1.4).

3. By means of (1.3) we can give an explicit formula for the residue of b_r. We recall the formula [3; p. 147]

$$\left(\frac{\log(1+t)}{t}\right)^z=z\sum_{m=0}^{\infty}\frac{t^m}{m!}\frac{B_m^{(m+z)}}{m+z}, \tag{3.1}$$

where $B_m^{(z)}$ is the Bernoulli number of order z defined by

$$\left(\frac{t}{e^t-1}\right)^z=\sum_{m=0}^{\infty}\frac{t^m}{m!}B_m^{(z)}.$$

In (3.1) take $z=\frac{1}{2}$ so that

$$\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r+1}t^r\right)^{1/2}=\sum_{m=0}^{\infty}\frac{t^m}{r!}\frac{B_r^{(r+1/2)}}{2r+1}.$$

Comparison with (1.3) leads at once to

$$b_r\equiv\frac{2\eta}{r!}\frac{B_r^{(r+1/2)}}{2r+1}\pmod{p},\qquad (0\leqq r\leqq\tfrac{1}{2}(p-1)). \tag{3.2}$$

4. To give a numerical illustration of (1.3) we take $p=13$. Then $Z=x+x^3+x^5$ and $V=3+9t+13t^2+11t^3+5t^4+t^5$, from which we get

$$\begin{aligned}V^2&\equiv 9+2t+3t^2+t^3+7t^4+5t^5+\cdots\\&\equiv -4(1-\tfrac{1}{2}t+\tfrac{1}{3}t^2-\tfrac{1}{5}t^3-\tfrac{1}{6}t^4+\tfrac{1}{8}t^5-\cdots)\pmod{13}.\end{aligned}$$

Similarly for $p=17$, we have $Z=x+x^2+x^3+2x^4+x^5+x^6+x^7$ and $V\equiv 8-2t-6t^2+6t^3+6t^4-6t^5+8t^6+t^7+\cdots\pmod{17}$, so that

$$\begin{aligned}V^2&\equiv -4+2t-7t^2+t^3+6t^4-5t^5-3t^6-8t^7+\cdots\\&\equiv -4(1-\tfrac{1}{2}t+\tfrac{1}{3}t^2-\tfrac{1}{4}t^3+\tfrac{1}{5}t^4-\tfrac{1}{6}t^5+\tfrac{1}{7}t^6-\tfrac{1}{8}t^7+\cdots)\pmod{17}.\end{aligned}$$

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THE FUNDAMENTAL THEOREM OF ALGEBRA

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The purpose of this note is to present a simple topological proof of the Fundamental Theorem of Algebra. This proof differs from that of Eilenberg and Niven [1] in that all constructions are carried out within the complex plane.

Notations and definitions. Z = the complex plane, z = arbitrary point of Z, $|z|$ = norm of z, $S=\{z \mid |z|=1\}$, $E=\{z \mid |z| \leqq 1\}$, and Int $E=\{z \mid |z|<1\}$. Observe that there is a homeomorphism h: Int $E \to Z$ which preserves amplitude (for example, $h(z)=z/(1-|z|)$).

LEMMA. *There is no continuous map f: $E \to S$ such that $f(z)=z^n$ for $z \in S$ (where n is an integer distinct from zero).*

Proof. Consider the diagram of homology groups and homomorphisms:

$$\begin{array}{ccc} H_2(E, S) & \xrightarrow{\partial} & H_1(S) \\ \downarrow f_*' & & \downarrow f_*'' \\ H_2(E, S) & \xrightarrow{\partial} & H_1(S). \end{array}$$

From elementary homology theory it is known that ∂ is an isomorphism, $f_*'=0$, and f_*'' is a homomorphism of degree n. Commutativity of the diagram forces $n=0$ and thus a contradiction.

THEOREM. *If f: $Z \to Z$ is continuous and satisfies*

$$\lim_{|z| \to \infty} f(z)/z^n = 1 \tag{1}$$

where n is an integer distinct from zero then $f(z)=0$ has a solution in Z.

Proof. Assume $f(z) \neq 0$ for all $z \in Z$. Define f_1: $Z \to S$ by $f_1(z)=f(z)/|f(z)|$. Define f_2: Int $E \to S$ by $f_2=f_1 h$. Condition (1) implies that f_2 may be extended to f_2: $E \to S$ by defining $f_2(z)=z^n$ for $z \in S$. This f_2 contradicts the lemma.

Remarks. The assumptions of the theorem are satisfied by a polynomial of degree n with leading coefficient 1; thus the Fundamental Theorem of Algebra is a consequence of this theorem.

The lemma yields for $n=1$ the classical result that S is not a retract of E. From this it follows in a well known manner that a continuous map f: $E \to E$ must have a fixed point.

References

1. Samuel Eilenberg and Ivan Niven, Bull. Amer. Math. Soc., vol. 50, 1944, pp. 246–248.
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IDEMPOTENT SEMIGROUPS

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An idempotent semigroup is a system of elements closed under an associative multiplication such that, for every element a of the semigroup, $a^2=a$. In this paper, I deal exclusively with idempotent semigroups.

For every element a of the system I define two sets, S_a and T_a, by the equivalences

$$ax = x \longleftrightarrow x \in S_a, \qquad xa = x \longleftrightarrow x \in T_a.$$

The sets S_a and T_a have the property that $S_a=S_b$ and $T_a=T_b \rightarrow a=b$. Thus the two sets characterize the element.

LEMMA 1. *An idempotent semigroup is anticommutative if and only if for every three elements a, b, and c of the semigroup $abc=ac$. A semigroup is said to be anticommutative when no two distinct elements of the semigroup commute.*

Proof: If we have $abc=ac$ for any three elements in the semigroup, then $(ab)(ba)=aba=a$, and $(ba)(ab)=bab=b$. Thus if $ab=ba$, then $a=b$. If the semigroup is anticommutative then, since $a(axa)=(axa)a=(axa)$, we must conclude that $a=axa$ for any x. Thus we have $abc=ab(cac)=[a(bc)a]c=a^2c=ac$.

LEMMA 2. *In an anticommutative idempotent semigroup, $S_a=S_{ab}$ and $T_b=T_{ab}$ for any a and b.*

Proof: Using Lemma 1, we see that since $ax=abx$, $ax=x \longleftrightarrow abx=x$ so $S_a=S_{ab}$. Since $xb=xab$, $xb=x \longleftrightarrow xab=x$ so $T_b=T_{ab}$.

The elements of an anticommutative idempotent semigroup are characterized by the two sets S_a and T_a and are multiplied by the rule that $(S_a, T_a)(S_b, T_b)=(S_a, T_b)$.

LEMMA 3. *In a commutative idempotent semigroup, $S_a=T_a$, and $S_{ab}=S_a \cap S_b$.*

Proof: The first part is obvious. Now if $x \in S_a \cap S_b$, then $abx=ax=x$ so that $x \in S_{ab}$, and so $S_a \cap S_b \subset S_{ab}$. If $x \in S_{ab}$, then $abx=x$. It follows that $ax=a(abx)=a^2bx=abx=x$, and $bx=b(abx)=ab^2x=abx=x$. Therefore $x \in S_a \cap S_b$, and $S_{ab} \subset S_a \cap S_b$. Q.e.d.

From the above lemma, we conclude that each element of a commutative idempotent semigroup is characterized by a single set, and that the multiplication is represented by the intersection.

Next I define two relations, R and L, by the equivalences

$$aRb \longleftrightarrow ab = b \quad \text{and} \quad ba = a$$
$$aLb \longleftrightarrow ab = a \quad \text{and} \quad ba = b.$$

LEMMA 4. *$aRb \longleftrightarrow S_a=S_b$, and $aLb \longleftrightarrow T_a=T_b$.* The proof is left to the reader. From Lemma 4, we see that the R and L relations are equivalence relations.

LEMMA 5. *$aRb \rightarrow caRcb$, and $aLb \rightarrow acLbc$.*

Proof: If aRb, then $ab=b$ and consequently $cab=cb$. It follows that $(ca)(cb)=(ca)cab=(ca)(ca)b=(ca)b=cab=cb$. Similarly $cbca=ca$. Therefore $caRcb$. In the same way, the second implication can be proved.

I define a relation P by the equivalence: $aba=a$ and $bab=b \leftrightarrow aPb$.

LEMMA 6. *aRb and $bLc \rightarrow aPc$.*

Proof: By lemma 5, $caRcb$ and $baLca$. By hypothesis, $ba=a$ and $cb=c$ so $caRc$ and $aLca$. Thus, by definition, $(ca)c=c$ and $a(ca)=a$. Consequently aPc.

Lemmas 7 and 8 are stated without proof.

LEMMA 7. *aLb or $aRb \rightarrow aPb$ and $caPcb$ and $acPbc$.*

LEMMA 8. *$aPb \rightarrow aRab$ and $bLab$ and $bRba$ and $aLba$.*

LEMMA 9. *Transitivity of the relation P: aPb and $bPc \rightarrow aPc$.*

Proof: If aPb and bPc, then by Lemma 8 $bLab$, $bLcb$, $bRba$, and $bRbc$. Since the R and L relations are symmetric and transitive, $abLcb$ and $baRbc$. By Lemma 5, $abaLcba$ and $cbaRcbc$. But $aba=a$ and $cbc=c$ so $aLcba$ and $cbaRc$. Thus by Lemma 6, aPc.

LEMMA 10. *$aPb \rightarrow caPcb$ and $acPbc$.*

Proof: If aPb, then by Lemma 8 $aRab$ and $bLab$. Now by Lemma 7, $caPcab$, $cbPcab$, $acPabc$, and $bcPabc$. By Lemma 9, $caPcb$ and $acPbc$.

LEMMA 11. *Let Q be a relation defining a homomorphism of an idempotent semigroup such that for all a and b in the semigroup $abQba$. Then $aPb \rightarrow aQb$.*

Proof: If aPb, then $(ab)(ba)=aba=a$, and $(ba)(ab)=bab=b$. But $(ab)(ba)Q(ba)(ab)$. Thus aQb.

THEOREM 1.* *Given an idempotent semigroup S, there exists a homomorphic mapping ϕ of S onto a commutative idempotent semigroup T such that the inverse image of any element of T is an anticommutative idempotent semigroup. The homomorphism ϕ is the weakest in the sense that any other commutative homomorphic image of S is also a homomorphic image of T.*

Proof: Lemmas 9 and 10 prove that the relation P defines a homomorphism. One easily verifies that $aoPba$ so the homomorphic image is commutative. Clearly a^2Pa, and by Lemmas 7 and 8 $aPb \rightarrow aPab$ so the equivalence classes (inverse images) are idempotent semigroups. Since $aba=a$ for any two elements in the same equivalence class, it follows from the proof of Lemma 1 that the equivalence classes are anticommutative idempotent semigroups. The last part

* This theorem is a special case of a more general theorem proved by A. H. Clifford; Semigroups admitting relative inverses, Ann. of Math., vol. 42, 1941, pp. 1037–1049.

of the theorem follows from Lemma 11.

Now I shall use the theory just developed to analyze finitely generated free idempotent semigroups. Given a set M of symbols (generators), a word is any finite sequence of generators. Under the operation of juxtaposition, the set of all words forms a semigroup F. If the set M has cardinal n, then F is called the free semigroup on n generators, and if n is finite then F is said to be finitely generated. Introducing the congruence relation $w^2=w$ for all words w, we obtain (defining multiplication in the obvious way) an idempotent semigroup S of congruence classes, each class representable by any word in it, and S is a homomorphic image of F under the mapping carrying each word of F into the element of S containing it; S is called the free idempotent semigroup on n generators. For each word w (element of F) I denote by C_w the set of distinct generators occurring in w. It is clear that if u and v are words lying in the same congruence class then $C_u=C_v$; hence, for each element a of S, there is a well-defined set C_a. It follows from the definition that $C_{ab}=C_a\cup C_b$.

LEMMA 12. $C_a=C_b \longleftrightarrow aPb$.

Proof: It is obvious that aPb implies $C_a=C_b$. The converse follows from Lemma 10 and the fact that $xyPyx$ for all x, y in S.

LEMMA 13. *$C_a=C_e$ and $C_b\subset C_a \rightarrow abc=ac$.*

Proof: Since $C_b\subset C_a$, $C_{ab}=C_a\cup C_b=C_a$. Hence $C_e=C_a=C_{ab}$, whence, by Lemma 12, $c(ab)c=c$. Therefore $(aca)bc=a[c(ab)c]=ac$. But, by Lemma 12, the hypothesis $C_a=C_e$ implies that $aca=a$. Hence $abc=ac$.

I define the length of a word to be the number of occurrences of (not necessarily distinct) generators in it; thus x and y being generators, the word x is of length 1, the words xx and xy are each of length 2, *etc.* Each word clearly has positive length; there is no need to introduce (as in the theory of free groups) an empty word. A word will be said to have minimum length if the element of S (congruence class in F) containing it contains no shorter word, and in what follows we shall assume at all times that the elements of S are represented by words of minimum length.

THEOREM 2.* *A finitely generated free idempotent semigroup is of finite order.*

LEMMA. *For any number n there is a length m such that all elements of minimum length m must be a product of at least n distinct generators.*

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* This theorem is considerably enhanced by the fact that there is an infinite sequence of three distinct symbols such that no two consecutive blocks of symbols are identical. See Marston Morse and Gustav Hedlund, Unending Chess, Symbolic Dynamics, and a Problem in Semigroups, Duke Mathematical Journal vol. 11, 1944, pp. 1–7. See also Green, J. A., and Rees, D., On semigroups in which $x^r=x$, Proc. Camb. Phil. Soc., vol. 48, 1952, pp. 35–40.

Proof: (by induction).

For $n=1$, $m=1$. Suppose the lemma is true for n, and consider the product of minimum length $2m+1$.

$$A = \prod_{i=1}^{2m+1} \alpha_i.$$

$$\text{Let } \prod_{i=1}^{m} \alpha_i = x \quad \text{and} \quad \prod_{i=m+2}^{2m+1} \alpha_i = y. \quad \text{Now} \quad A = x\alpha_{m+1}y.$$

By the induction hypothesis C_x and C_y each contains at least n distinct generators. Clearly C_x and C_y are contained in C_A. Now if C_A contains only n distinct generators, then $C_x = C_y = C_A$. Also $\alpha_{m+1} \in C_x$ and $\alpha_{m+1} \in C_y$. Thus by Lemma 13, $A = x\alpha_{m+1}\, y = xy$. But xy is of minimum length $2m$ or less contrary to the hypothesis that A have minimum length $2m+1$. Thus all expressions of minimum length $2m+1$ must be a product of at least $n+1$ distinct generators. This proves the lemma. It follows from the lemma that all elements of minimum length generated by a finite number of generators must not exceed a certain length, and hence there must be only a finite number of them.

COROLLARY. *All finitely generated idempotent semigroups are of finite order.*

By developing the theory of free idempotent semigroups, one arrives at the following formula for the order I_N of the free idempotent semigroup on N generators.

$$I_N = \sum_{r=1}^{N} C_r^N \prod_{i=1}^{r} (r - i + 1)^{2^i}.$$

CLASSROOM NOTES

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A PAPER MODEL OF THE PROJECTIVE PLANE

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The study and classification of two-dimensional closed manifolds which takes place in elementary topology leads to the consideration of the cross-cap or its topological equivalent the Möbius strip. A simple discussion of the fact that the projective plane is topologically equivalent to a sphere with a hole covered

by a cross-cap is found in [1]. Here we give the construction of a paper model of the projective plane.

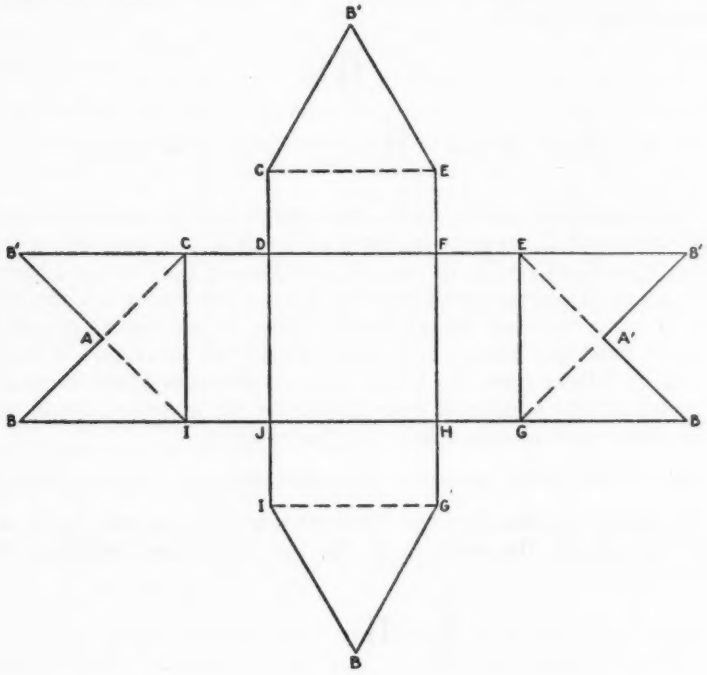

FIG. 1

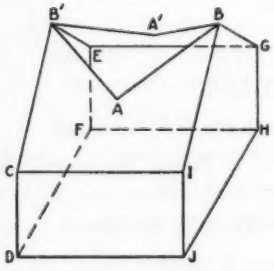

FIG. 2

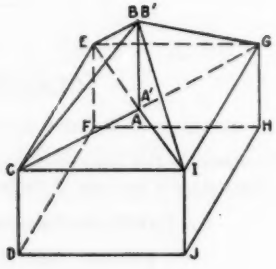

FIG. 3

A diagram of the surface as cut from a sheet of paper is given in Figure 1. The segments BI, BG, $B'E$, $B'C$, CE, EG, GI, IC are each equal to a side of the square $DFHJ$. Also CD, EF, HG, IJ are equal in length. When upward folds

are made along segments CI, DJ, DF, HJ, HF, GE and all the segments with the same labels identified, the surface in Figure 2 is obtained with the hole $ABA'B'$. If also downward folds are made along AC, AI, $A'E$, $A'G$ and upward folds are made along CE and IG in Figure 1, then Figure 2 can be carried over into the form of Figure 3 by identifying B with B' and A with A'. This identification matches the opposite pairs of points of the hole $ABA'B'$ to give the closed surface in Figure 3 with only the segment AB identified with the segment $A'B'$ and the segment BA' identified with $B'A$. Thus Figure 2 gives a surface homeomorphic to a sphere with a hole, and Figure 3 gives a closed surface homeomorphic to a sphere with a cross-cap, that is, homeomorphic to the projective plane. Of course, the surface of Figure 3 appears to cross itself; however, we must think of the segments AB and $A'B$ as distinct sets of points having only the corresponding endpoints identified.

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ON NUMERICAL SOLUTIONS TO THE ONE-DIMENSIONAL WAVE EQUATION

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In chapter 3, sections 3.21–3.22 of F. B. Hildebrand's *Methods of Applied Mathematics* one finds a discussion of a method of finite difference approximation for obtaining numerical solutions to the one-dimensional wave equation:

$$v^2 \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = 0. \tag{1}$$

In the discussion, however, it has not been pointed out that Equation 279 on page 326 is not just an approximation, but in fact is exactly true. Indeed a more general equality is

$$\begin{aligned} \phi(x + vh, t + H) = {} & \phi(x + vH, t + h) + \phi(x - vH, t - h) \\ & - \phi(x - vh, t - H). \end{aligned} \tag{2}$$

When $H=0$ this reduces to Hildebrand's Equation 279.

The general solution to (1) is

$$\phi(x, t) = f(x + vt) + g(x - vt) \tag{3}$$

where f and g are arbitrary functions whose second partial derivatives with respect to x and t exist. It is easily verified that (2) is an identity if (3) holds.

SKEW-SYMMETRIC DETERMINANTS

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To show that a skew-symmetric determinant of even order is a perfect square, Burnside and Panton (*Theory of Equations*, vol. II) have proved the result for fourth order and then generalised the result by a wave of the hand. I feel that the proof would have been more elegant if they had started from a higher order and by successive reduction established the result. The object of this note is to give such a proof.

Let n be an even number, let

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ a_{31} & a_{32} \cdots a_{3n} \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix}$$

and let Δ_{n-2} be the minor of

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad \text{in } \Delta_n$$

Taking A_{11}, A_{12} *etc.* to be the cofactors of a_{11}, a_{12} *etc.* in Δ_n we have

$$\begin{vmatrix} 1 & 0 & 0 & \cdots 0 \\ 0 & 1 & 0 & \cdots 0 \\ a_{31} & a_{32} & a_{33} \cdots & a_{3n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} \cdots & a_{nn} \end{vmatrix} \times \begin{vmatrix} A_{11} & A_{21} \cdots A_{n1} \\ A_{12} & A_{22} \cdots A_{n2} \\ A_{13} & A_{23} \cdots A_{n3} \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ A_{1n} & A_{2n} \cdots A_{nn} \end{vmatrix} = \begin{vmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{vmatrix} \cdot \Delta_n^{n-2};$$

that is

$$\Delta_{n-2} \cdot \Delta_n^{n-1} = \begin{vmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{vmatrix} \cdot \Delta_n^{n-2}$$

or

$$\Delta_{n-2} \cdot \Delta_n = -A_{21} \cdot A_{12} = (A_{12})^2,$$

since $A_{rr}=0$ (being a skew-symmetric determinant of odd order) and $A_{rs} = (-1)^{n-1}A_{sr} = -A_{sr}$ [changing rows into columns and multiplying by $(-1)^{n-1}$ we get the result]. Therefore Δ_n is a perfect square if Δ_{n-2} is a perfect square. But Δ_2 is known to be a perfect square. Hence the result.

ON THE MOTION OF A PARTICLE THROUGH A RESISTING MEDIUM OF VARIABLE DENSITY

EUGENE LEIMANIS, University of British Columbia

The motion of a particle through a resisting medium of constant density is treated in most of the well-known textbooks of mechanics, as, for example, Sommerfeld [1]. However, as R. H. Bacon [2] has pointed out, many interesting problems involve motion of a particle through a resisting medium whose density is a function of the position of the particle. He considers, in particular, the fall of a particle dropped from rest from a great height through an atmosphere whose density decreases exponentially with the altitude. Take the positive z-direction as vertically downward, and let the particle be dropped at the instant $t=0$ from the height $z=H$ with initial velocity $dz/dt=0$. Further assume that the air resistance is proportional to the product of the air density and the nth power of the instantaneous speed. Then the equation of motion of the falling particle is

$$\frac{d^2z}{dt^2} = g - ke^{az}\left(\frac{dz}{dt}\right)^n, \tag{1}$$

where g denotes the acceleration of gravity. Bacon inquires if it is possible to obtain for the cases $n=1$ and $n=2$ solutions of this equation in closed form. This question has been answered in the affirmative by Sexl [3], who gave explicit formulas for the solutions.

An obvious generalization of the equation (1) is

$$\frac{d^2z}{dt^2} = g - f(z)\left(\frac{dz}{dt}\right)^n, \tag{2}$$

where $f(z)$ is any function of z different from zero. The aim of this note is to indicate some values of the exponent n and some forms of the function $f(z)$ for which equation (2) is integrable by quadratures.

Let

$$\frac{dz}{dt} = p,$$

then equation (2) may be written in the form

$$p\frac{dp}{dz} = g - f(z)p^n.$$

By a further substitution $p=u^{-1/2}$, the last equation goes over into

$$\frac{du}{dz} = -2gu^2 + 2f(z)u^{2-n/2}. \tag{3}$$

We will be concerned with the integration of this equation in the following

cases: $2-n/2=0, 1, 3/2, 2, 3$, *i.e.*, $n=4, 2, 1, 0, -2$ respectively.

Case (*i*): $n=4$. Equation (3) then reads

$$\frac{du}{dz} + 2gu^2 = 2f(z),$$

which is a generalized Riccati equation. If $f(z)=kz^m$, we have the original equation of Riccati

$$\frac{du}{dz} + 2gu^2 = 2kz^m \qquad (k \text{ constant}),$$

which is soluble by means of algebraic, exponential, and logarithmic functions for all values of m given by the formula

$$m = -\frac{4s}{2s+1},$$

where s is zero or a positive integer. In the limiting case, when $s\to\infty$, the value to which the exponent tends is -2, and the equation is still soluble by the use of elementary functions.

Case (*ii*): $n=2$. Equation (3) then becomes

$$\frac{du}{dz} = -2gu^2 + 2f(z)u,$$

which is a Bernoulli equation. By reduction to a linear equation it is easily integrable by quadratures for any given $f(z)$. Sexl [3] has considered, in particular, the case $f(z)=ke^{az}$.

Case (*iii*): $n=1$. Equation (3) is then of the form

$$\frac{du}{dz} = -2gu^2 + 2f(z)u^{3/2}.$$

When we make the substitution $u=y^2$, the equation becomes

$$\frac{dy}{dz} = f(z)y^2 - gy^3,$$

or, by taking $Z=-gz$ and $Y=-y$ as new variables, we obtain

$$\frac{dY}{dZ} = \phi(Z)Y^2 + Y^3, \tag{4}$$

where

$$\phi(Z) = \frac{1}{g} f\left(-\frac{Z}{g}\right).$$

According to Appell [4] the simplest cases in which equation (4) can be solved in finite terms are:

$$\text{(a)}\quad \phi(Z) = \frac{k}{\sqrt{Z}}, \qquad \text{(b)}\quad \phi(Z) = ke^{Z}, \qquad \text{(c)}\quad \phi(Z) = kZ, \qquad \text{(d)}\quad \phi(Z) = \frac{k}{Z^2},$$

where, in general, Z may also be replaced by a linear function of Z. The case $f(z) = ke^{az}$, considered by Sexl [3], can be easily reduced to the case (b), if one chooses new variables defined by the formulas

$$-\frac{\alpha}{g}Z = Z_1, \qquad Y = \sqrt{-\frac{\alpha}{g}}\,Y_1.$$

The cases (b) and (c) when the density of the atmosphere decreases exponentially or linearly with the altitude may have some interest in practical applications.

Case (iv): $n = 0$. Equation (3) is then integrable by separation of variables for any $f(z)$.

Case (v): $n = -2$. Equation (3) becomes in this case

$$\frac{du}{dz} = -2gu^2 + 2f(z)u^3. \tag{5}$$

Introduce instead of z a new independent variable Z defined by the formula

$$\frac{dZ}{dz} = 2f(z).$$

Then for $f(z) = z$, $1/z$, $1/\sqrt{z}$, $1/z^2$ the equation (5) goes over into the equation

$$\frac{du}{dZ} = \phi(Z)u^2 + u^3,$$

where $\phi(Z)$ is of the form (a), (b), (c), (d) respectively.

The integration of equation (5) when $g = g(z)$ is a function of z and $f(z) = ke^{az}$ has been considered by the author in a recent paper [5].

References

1. Sommerfeld, A., Lectures on Theoretical Physics, vol. 1, Mechanics. New York, Academic Press Inc., 1952, p. 21.

2. Bacon, R. H., Motion of a particle through a resisting medium of variable density, Amer. J. of Phys., vol. 19, p. 64, 1951.

3. Sexl, Th., Über die Bewegung eines Massenpunktes in einem widerstehenden Mittel veränderlicher Dichte, Acta Phys. Austriaca, vol. 5, pp. 148–151, 1951.

4. Appell, P., Sur les invariants de quelques équations différentielles, Journal Math. Pures Appl. (4) vol. 5, pp. 361–423, 1889.

5. Leimanis, E. Über die Bewegung eines Massenpunktes in einem widerstehenden Mittel veränderlicher Dichte, Acta Phys. Austriaca, vol. 6, pp. 319–321, 1953.

DIFFERENTIATING THE LOGARITHM

R. C. YATES, U. S. Military Academy, West Point

In differentiating $\ln x$ by the "delta process" a point is reached where it is necessary to introduce a carefully selected coefficient—an act which, to the student, seems quite artificial. If, instead, the function $x \cdot \ln x$ be considered, the coefficient enters naturally. Thus, omitting some details,

$$y = x \cdot \ln x$$

$$\frac{\Delta y}{\Delta x} = \frac{x}{\Delta x} \ln\left(1 + \frac{\Delta x}{x}\right) + \ln(x + \Delta x)$$

$$Dy = 1 + \ln x.$$

An important by-product occurs if the function $x \cdot \ln x$ be differentiated formally as a product:

$$Dy = x \cdot D(\ln x) + \ln x$$

and the two results compared. Thus

$$D(\ln x) = \frac{1}{x}.$$

DERIVATIVES OF IMPLICIT FUNCTIONS

M. R. SPIEGEL, Rensselaer Polytechnic Institute

When presenting a definition of the derivative of a function of a single variable, most textbooks and teachers place emphasis on derivatives of explicit functions using the "Δ-process." Thus the student is informed: Given $y = f(x)$, the derivative of y with respect to x is given by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

if such a limit exists. After this follows a tremendous number of problems in which the student has to write $dy/dx = \cdots$ a large portion of the time. It seems small wonder then that when presented with an implicit function of x, such as that defined by the relation $4x^2 + y^3 - y = 6x$, the student starts off by writing $dy/dx = \cdots$ and then proceeds to differentiate each term (I trust that many teachers have found the same to be true with their students). In seeking to abolish this sort of thing and to see whether I could instill a better understanding of derivatives of implicit functions, I found that it was instructive to point out to the student, by a few examples, that derivatives of implicit functions need not depend (as many of them believed) on theorems already proved, but that the "Δ-process" could be used directly.

To see how natural such an approach appears to the student, let us consider the following simple example:

If $4x^2+y^3-y=6x$ defines y as a function of x find dy/dx by using the "Δ-process."

Solution: Proceeding according to the "Δ-process" we give x an increment Δx and y "takes on" the increment Δy in such a way that $x+\Delta x$ and $y+\Delta y$ satisfy the given relation. Thus

$$4(x+\Delta x)^2+(y+\Delta y)^3-(y+\Delta y)=6(x+\Delta x)$$

or

$$4x^2+8x\Delta x+4(\Delta x)^2+y^3+3y^2\Delta y+3y(\Delta y)^2+(\Delta y)^3-y-\Delta y = 6x+6\Delta x. \tag{1}$$

Subtracting

$$4x^2+y^3-y=6x$$

from (1) we obtain

$$8x\Delta x+4(\Delta x)^2+3y^2\Delta y+3y(\Delta y)^2+(\Delta y)^3-\Delta y=6\Delta x. \tag{2}$$

Upon division of both sides of equation (2) by Δx we have

$$8x+4\Delta x+3y^2\frac{\Delta y}{\Delta x}+3y\left(\frac{\Delta y}{\Delta x}\right)\Delta y+\left(\frac{\Delta y}{\Delta x}\right)(\Delta y)^2-\frac{\Delta y}{\Delta x}=6.$$

Taking the limit as $\Delta x\to 0$, assuming that when $\Delta x\to 0$, $\Delta y\to 0$ and $(\Delta y/\Delta x)\to(dy/dx)$, we find

$$8x+3y^2\frac{dy}{dx}-\frac{dy}{dx}=6$$

from which finally

$$\frac{dy}{dx}=\frac{6-8x}{3y^2-1}.$$

Similar approaches adopted when dealing with partial derivatives seem to give the student better understanding of explicit and implicit functions of more than one variable. Furthermore the theory of functions of a single variable (explicit and implicit) and the theory of functions of two or more variables seem to become more unified in the minds of the students.

A FUNDAMENTAL THEOREM OF THE CALCULUS

ISRAEL HALPERIN, Queen's University

Instructors of undergraduate calculus courses frequently use a non-rigorous geometric argument to "prove" that a function $f(x)$ must be constant on an interval if it has a derivative equal to zero at each point of the interval. The usual rigorous proof of this theorem uses the theorem of the mean which is itself based on the real variable theorem that a continuous function on a closed finite interval attains its maximum and minimum values. The difficulty, for the instructor, is that these deeper theorems are not known to the student.

However a completely rigorous proof can be derived at once from the following *weakened* form of the theorem of the mean:

THEOREM W. *If $f(x)$ has a derivative $f'(x)$ for each $a \leqq x \leqq b$ then for some $a \leqq x_0 \leqq b$*

$$|f'(x_0)| \geqq \left|\frac{f(b) - f(a)}{b - a}\right|.$$

The following proof of Theorem W is simple enough for any undergraduate calculus course!

Without loss of generality, suppose the interval (a, b) is $(0, 1)$. Denote $|f(1)-f(0)|$ by k and let p_1 be the least of the integers $0, 1, \cdots, 9$ for which

$$\left|\frac{f(0.(p_1+1)) - f(0.p_1)}{1/10}\right| \geqq k.$$

There is such a p_1, for if there were not it would follow that

$$|f(1) - f(0)| \leqq |f(1) - f(0.9)| + |f(0.9) - f(0.8)| + \cdots + |f(0.1) - f(0)|$$
$$< \frac{k}{10} + \frac{k}{10} + \cdots + \frac{k}{10} = k,$$

that is, $|f(1)-f(0)| < k$, a contradiction. Designate the interval $(0.p_1, 0.(p_1+1))$ as (a_1, b_1).

Similarly let p_2 be the least of $0, 1, \cdots, 9$ for which

$$\left|\frac{f(0.p_1(p_2+1)) - f(0.p_1p_2)}{1/100}\right| \geqq k$$

and designate $(0.p_1p_2, 0.p_1(p_2+1))$ as (a_2, b_2).

Repetition of this procedure yields a number x_0, given by its decimal expansion

$$x_0 = 0.p_1p_2\cdots$$

and a sequence of intervals (a_n, b_n) such that for every n

$$\left|\frac{f(b_n) - f(a_n)}{b_n - a_n}\right| \geqq k$$

and x_0 is contained in, or is an end point of, the interval (a_n, b_n) and $b_n - a_n \to 0$ as $n \to \infty$. If x_0 is contained in a particular (a_n, b_n) we may preserve the inequality either by replacing a_n by x_0 or by replacing b_n by x_0. Then it follows from the definition of derivative that $|f'(x_0)| \geqq k$, as stated.

The sophisticated student will see that this argument is valid for functions $f(x)$ valued in an arbitrary linear, normed space. He will also use the same argument to prove (as variations of Theorem W) that for *real valued* $f(x)$ there are suitable x_1, x_2 on (a, b) with

$$f'(x_1) \leqq \frac{f(b) - f(a)}{b - a} \quad \text{and} \quad f'(x_2) \geqq \frac{f(b) - f(a)}{b - a}$$

respectively (actually, the real-variable proof of the theorem of the mean shows that there is an ξ in (a, b) for which *both* of these inequalities hold simultaneously).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1101. *Proposed by M. S. Webster, Purdue University*

The inequality of Bernoulli is often stated as follows: If $h > -1$ and $h \neq 0$, then $(1+h)^n > 1+nh$, where n is an integer greater than unity. What is the generalization if n is an arbitrary real number?

E 1102. *Proposed by Harley Flanders, University of California*

Let C be an $n \times n$ matrix such that whenever $C = AB$ then $C = BA$. What is C?

E 1103. *Proposed by Paul Monsky, Brooklyn, N. Y.*

Find the locus of the vertex of a tri-rectangular trihedral angle which moves

so that its edges intersect a given circle.

E 1104. *Proposed by J. R. Hatcher, Fisk University*

Let $g(x)$ be a given continuous function satisfying $\int_\alpha^\beta g(x)dx = 0$. Find $f(x)$ such that $\int_\alpha^\beta f^2(x)dx$ is a minimum subject to the conditions $\int_\alpha^\beta f(x)dx = \gamma$ and $\int_\alpha^\beta f(x)g(x)dx = \delta$, where $\alpha, \beta, \gamma, \delta$ are constants.

E 1105. *Proposed by R. M. Mason, Washington, D. C.*

"He had a line or two rigged up to help him cross the widest spaces—Long John's earrings, they were called: and he would hand himself from one place to another, now using the crutch, now trailing it alongside by the lanyard, as quickly as another man could walk."—Robert Louis Stevenson, *Treasure Island*.

Find the probability that on the nth step Silver is not supported by his crutch.

SOLUTIONS

Value of a Derivative

E 1071 [1953, 417]. *Proposed by Albert Wilansky, Lehigh University*

If $f(x) = \int_0^x \cos(1/t)dt$, show that $f'(0) = 0$.

Solution by George Piranian, University of Michigan. Since $f(0) = 0$, $f'(0) = \lim_{x \to 0} x^{-1}f(x)$. Also, since

$$f(x) = -x^2 \sin(1/x) + \int_0^x 2t \sin(1/t)dt,$$

$|f(x)| \leqq 2x^2$, and the result follows.

Also solved by P. M. Anselone, Richard Courter, N. J. Fine, Donald Greenspan, M. S. Klamkin, S. Leja, A. E. Livingston, C. S. Ogilvy, S. Parameswaran, G. B. Parrish, L. L. Pennisi, M. Perisastri, L. A. Ringenberg, O. E. Stanaitis, and the proposer.

A Four-Digit Number

E 1072 [1953, 417]. *Proposed by S. J. Jasper, East Tennessee State College*

Find a four-digit number in base ten which, when the order of the digits is reversed, becomes an equivalent number in base seven.

Solution by Julian Braun, Washington, D. C. Let the number in the base ten appear as $ABCD$. Then

$$1000A + 100B + 10C + D = 343D + 49C + 7B + A,$$

whence

$$38D - 111A = (31B - 13C)/3.$$

Because $B, C \leqq 6$, the values of (A, D) are limited to (1, 2), (1, 3), (2, 5), and (2, 6), which give values for $38D-111A$ of -35, 3, -32, and 6, respectively. There are only seventeen pairs (B, C) which yield values of $(31B-13C)/3$ which are integers. Only in the case $(B, C)=(1, 1)$ do we obtain a value of $(31B-13C)/3$ matching a value of $38D-111A$ given above, namely the case for $(A, D)=(2, 6)$. Thus the *unique* four-digit number in the base ten satisfying the given requirement is 2116.

Also solved by A. N. Aheart, Norman Anning, Leon Bankoff, Charles Berndtson and W. S. Wilf (jointly), W. E. Briggs, W. E. Buker, P. L. Chessin, J. E. D'Atri, Monte Dernham, Fred Discepoli, Mildred Going, H. W. Gould, Vern Hoggatt, Douglas Holdridge, A. R. Hyde, John Jones, Jr., Allen Kirchberg, M. Kitamura, M. S. Klamkin, Sidney Kravitz, D. D. McCracken, Sylvia Manheim, Fred Marer, D. C. B. Marsh, George Millman, Sharon Murnick, E. J. Musch, J. B. Muskat, E. A. Nordhaus, C. S. Ogilvy, Margaret Olmsted, S. Parameswaran, G. B. Parrish, M. J. Pascual, L. A. Ringenberg, H. A. Robinson, R. Shafer, L. J. Shiflett, J. D. Thomas, C. W. Trigg, W. R. Van Voorhis, R. Z. Vause, J. W. Young, and the proposer.

Trigg remarked that the above is the only solution for bases x and y where $10 \geqq x > y$. Dernham pointed out the following curiosity: The corresponding problem for two-digit numbers yields $23_{10}=32_7$. Doubling each side we get $46_{10}=64_7$. There are no other two-digit solutions, and there are no three-digit solutions. Squaring each side of the last equality we find $2116_{10}=6112_7$.

Moment of Inertia of a System

E 1074 [1953, 417]. *Proposed by Vern Hoggatt, Oregon State College*

A regular n-gon with the center O has a particle of mass m at each vertex. Let PO be a segment perpendicular to the plane of the n-gon and let l be a line through P parallel to the plane of the n-gon. Show that the sum of the moments of inertia of the n masses about the line l is independent of the (restricted) orientation of l.

Solution by J. F. Burke, Lehigh University. Let I be the moment of inertia of the system with respect to l, and let r be the radius of the n-gon. Through O draw a line l_x parallel to l and assume that this line makes an angle ϕ with some radius of the n-gon. Then

$$I = \sum_{k=0}^{n-1} mr^2 \sin^2(\phi + 2k\pi/n) + mn\overline{OP}^2$$

$$= mr^2(n/2) + mn\overline{OP}^2.$$

Also solved by A. G. Grace and W. J. Klimczak (jointly), Douglas Holdridge, A. R. Hyde, M. S. Klamkin, A. E. Livingston, Beckham Martin, S. Parameswaran, G. B. Parrish, M. Perisastri, L. A. Ringenberg, O. E. Stanaitis, C. A. Swanson, J. V. Whittaker, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4573. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Show that, if $p=\log(\sqrt{2}+1)$,

$$\int_0^p \frac{x \sinh x}{\sqrt{1-\sinh^2 x}}\,dx = \frac{\pi}{4}\log 2.$$

4574. *Proposed by H. S. Shapiro, Bell Telephone Laboratories, Murray Hill, N. J.*

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be the power series of a rational function, and let the a_n be rational integers. If $a_n=0$ for $n=(k!+k)$, $k=1, 2, \cdots$, then f is a polynomial.

4575. *Proposed by the late Joseph Rosenbaum, Hartford, Connecticut*

Find the complete solution in positive integers of the Diophantine system,

$$2uv - xy = 16, \qquad xv - uy = 12.$$

4576. *Proposed by G. A. Garreau, Northampton Polytechnic College, London, England*

Construct an infinite matrix A such that A^{m-1} exists, whereas A^m does not.

4577. *Proposed by Albert Wilansky, Lehigh University*

Given a point x in a Hausdorff space, does there exist a set of neighborhoods of x whose intersection contains only x, and such that of any two neighborhoods in the set one includes the other (*i.e.* the neighborhoods are "nested")?

SOLUTIONS

Deficient Numbers

4507 [1952, 555]. *Proposed by Leo Moser, University of Alberta, Canada.*

In L. E. Dickson, *History of the Theory of Numbers*, we find the statement: Charles de Neuvegelise proved that the products $3\cdot 4, \cdots, 8\cdot 9$ of two consecutive numbers are abundant.

Prove that there are infinitely many numbers of the form $a(a+1)$ which are deficient.

Solution by Bryant Tuckerman, Oberlin College and Institute for Advanced Study. Let p_i be the consecutive primes ($p_2=3$); let $b_n=4\prod_{i=2}^{i=n} p_i$, ($n\geqq 2$); and let $c_n=(b_n-2)(b_n-1)$. Then c_n has the form $a(a+1)$, and will be shown to be deficient for all sufficiently large n.

None of $p_2, \cdots, p_n$ divides b_n-2 or b_n-1, and b_n-2 is not a multiple of 4. Hence $c_n=2\prod_{j=1}^{j=k} q_j^{d_j}$ where $k>0$, $d_j>0$, and the q_j are certain distinct primes each greater than p_n. Furthermore, $k<2n+1$ (for otherwise, $c_n>2p_n^k\geqq 2p_n^{2n+1}>2^4p_n^{2n-2}\geqq b_n^2>c_n$, a contradiction).

The ratio of the sum of the divisors of c_n to c_n is

$$\frac{\sigma(c_n)}{c_n}=\frac{3}{2}\prod_{j=1}^{k}\frac{q_j^{d_j+1}-1}{(q_j-1)q_j^{d_j}}<\frac{3}{2}\prod_{j=1}^{k}\frac{q_j}{q_j-1}<\frac{3}{2}\left(\frac{p_n}{p_n-1}\right)^k<\frac{3}{2}\left(\frac{p_n}{p_n-1}\right)^{2n+1}$$

$$=\frac{3}{2}\left[\left(1+\frac{1}{p_n-1}\right)^{p_n-1}\right]^{(2n+1)/(p_n-1)},$$

which approaches 3/2, since $p_n\to\infty$ and $n/p_n\to 0$ as $n\to\infty$. Hence for some N, and for all $n>N$, $\sigma(c_n)/c_n<2$, so that c_n is deficient.

Also solved by Fritz Herzog and the Proposer.

Editorial Note. Herzog also remarks that it can be shown similarly that for fixed k, infinitely many of the binomial coefficients $\binom{n}{k}$ are deficient.

A Polynomial Assuming Prescribed Values

4508 [1952, 640]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

Find a polynomial $F(x)$ of lowest degree such that $F(x)+\omega^s$ is divisible by $(x-\omega^s)^r$ for $s=0, 1, 2, \cdots, p-1$, where ω is a primitive pth root of unity and r is a given positive integer. (This generalizes a problem in Goursat-Hedrick, *A Course in Mathematical Analysis*, vol. 1, p. 32.)

Solution by Chih-yi Wang, University of Minnesota. Since $F'(x)$ is divisible by $(x^p-1)^{r-1}$, and the binomials $x-\omega^s$ are relatively prime to one another, it is evident that the degree of the polynomial $F'(x)$ cannot be lower than $p(r-1)$ and hence that of $F(x)$ cannot be lower than $p(r-1)+1$. Let

$$F'(x)=a(x^p-1)^{r-1},$$

where $a(\neq 0)$ is some constant which has to be determined. By integrating from 0 to x we get

$$F(x)-F(0)=a\int_0^x (t^p-1)^{r-1}dt.$$

But $F(\omega^s)+\omega^s=0$ for $s=0, 1, \cdots, p-1$, so that

$$-\omega^s - F(0) = \int_0^{\omega^s} a(t^p - 1)^{r-1}dt$$

$$= \omega^s\left\{a\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{1}{p[(r-1)-k]+1}\right\}$$

$$= \omega^s\left\{a\int_0^1 (t^p - 1)^{r-1}dt\right\}, \qquad s = 0, 1, \cdots, p-1.$$

These p equations are satisfied if and only if

$$F(0) = 0, \qquad a = -1\Big/\int_0^1 (t^p - 1)^{r-1}dt.$$

Hence the required polynomial is

$$F(x) = -\int_0^x (t^p - 1)^{r-1}dt\Big/\int_0^1 (t^p - 1)^{r-1}dt.$$

Also solved by C. N. Campopiano, Harley Flanders, M. S. Klamkin, Norman Miller, and the Proposer.

A Multiplicative Function

4509 [1952, 640]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Prove that the most general multiplicative function $f(n)$ such that $f(n+1) \sim f(n)$ is $f(n) = n^a$ for some constant a.

Solution by A. E. Livingston, University of Washington. Since $f(x)$ is multiplicative, we have $f(xy) = f(x)f(y)$ for all real x, y. We note that $[f(0)]^2 = f(0)$ and, hence, that $f(0) = 0$ or $f(0) = 1$. If $f(0) = 1$, then we have $1 = f(0) = f(0)f(x) = f(x)$ for all x. In what follows, then, we take $f(0) = 0$. Furthermore it is clear that $f(1) = 1$. Hence $[f(-1)]^2 = f(1) = 1$, so that $f(-1)$ is either 1 or -1. There are thus two solutions when $f(0) = 0$: either $f(x)$ is an odd function of x or $f(x)$ is even. Consequently, we need only consider the problem for $x > 0$.

We have

$$f(x+1) = f(x)f\left(1 + \frac{1}{x}\right), \qquad f(x) = f(x+1)f\left(1 - \frac{1}{x+1}\right).$$

Since $f(1) = 1$ and $f(x+1)/f(x) \to 1$ as $x \to \infty$, we have $f(1^-) = f(1^+) = f(1)$, from which we deduce in the usual way that $f(x)$ is continuous for all x.

Since $f(x)$ is continuous, we have $f(x^b) = [f(x)]^b$ for every real b. In particular, if $b = \log c$, $c > 0$ and $c \neq 1$, then

$$[f(x)]^{\log c} = f(x^{\log c}) \equiv f(c^{\log x}) = [f(c)]^{\log x}.$$

Therefore

$$f(x) = \left\{ [f(c)]^{1/\log c} \right\}^{\log x} = x^{\log f(c)/\log c}$$

which equals x^a, say.

Also solved by Harry Furstenberg, M. S. Klamkin, Albert Wilansky, and the Proposer.

A Monotone Non-increasing Sequence

4510 [1952, 640]. *Proposed by H. P. Thielman, Iowa State College*

If $a_1, a_2, a_3, \cdots$ is a sequence of real numbers such that $a_{n+1} = 1/n^p a_n$, where $p > 0$ and $n = 1, 2, 3, \cdots$, prove that the sequence $\{a_n\}$ is monotone non-increasing if and only if $a_1 = (\pi/2)^{p/2}$.

Solution by R. H. Boyer, Carnegie Institute of Technology, Pittsburgh. Using the recurrence relation we find

$$a_{2n+1} = \left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}\right)^p a_1, \qquad \frac{1}{a_{2n}} = \left(\frac{3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots (2n-2)}\right)^p a_1.$$

$\{a_n\}$ will be monotone non-increasing if and only if $a_{2n+1} \leqq a_{2n} \leqq a_{2n-1}$ for all n. Thus

$$\left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}\right)^p a_1 \leqq \left(\frac{2 \cdot 4 \cdots (2n-2)}{3 \cdot 5 \cdots (2n-1)}\right)^p \frac{1}{a_1} \leqq \left(\frac{1 \cdot 3 \cdots (2n-3)}{2 \cdot 4 \cdots (2n-2)}\right)^p a_1,$$

or

$$\left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n-2}{2n-1} \cdot \frac{2n}{2n-1}\right) \geqq a_1^{2/p} \geqq \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n-2}{2n-3} \cdot \frac{2n-2}{2n-1}\right).$$

But we know from Wallis' formula that, for all n,

$$\left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n-2}{2n-1} \cdot \frac{2n}{2n-1}\right) \geqq \frac{\pi}{2} \geqq \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n-2}{2n-3} \cdot \frac{2n-2}{2n-1}\right).$$

It is therefore necessary and sufficient that $a_1 = (\pi/2)^{p/2}$.

Also solved by Harley Flanders, Harry Furstenberg, C. V. Gregg, Emil Grosswald, P. G. Kirmser, Norman Miller, S. Parameswaran, R. F. Reeves, A. J. Rodriguez, O. E. Stanaitis, Chih-yi Wang, J. V. Whittaker, J. E. Wilkins, Jr., and the Proposer.

A Diophantine Equation

4511 [1952, 640]. *Proposed by W. V. Parker, Alabama Polytechnic Institute*

Solve the Diophantine equation

$$x^2 + y^2 + (xy)^2 = z^2.$$

Solution by C. V. Gregg, Huddersfield Technical College, England.

I. It is easily verified (and was noted by Diophantus) that solutions are given by the following for any positive integer x:

$$y = 0, \quad z = x; \qquad y = x \pm 1, \quad z = x^2 \pm x + 1.$$

II. If x is held fixed the given equation becomes a Pell-Fermat equation in y and z. Hence if (x, y_0, z_0) is a solution and (y_0, z_0), (y_1, z_1), $\cdots$, is a sequence of number pairs such that

$$(1) \qquad y_{n+1} = (2x^2 + 1)y_n + 2xz_n, \qquad z_{n+1} = 2x(x^2 + 1)y_n + (2x^2 + 1)z_n,$$

then (x, y_n, z_n) is a solution for all n. The sequence can be extended backward by

$$(2) \qquad y_{n-1} = (2x^2 + 1)y_n - 2xz_n, \qquad z_{n-1} = -2x(x^2 + 1)y_n + (2x^2 + 1)z_n.$$

For each arbitrary x there are thus three infinite sequences of solutions obtained by (1) from **I**. However, the last two may be shown to be the same sequence in reverse order with the sign of y_n changed. Furthermore, each solution found by developing these sequences can, by interchanging the values of x and y, initiate a new sequence, and so on.

III. It will be shown that, by using the procedures of **II**, all possible solutions of the given equation may be obtained from **I**, in fact, may be obtained from solutions $(0, N, N)$ or $(N, 0, N)$ where N is an integer. Let (x_0, y_0, z_0) be any solution in positive integers in which $y_0 > x_0$. Then

$$z_0^2 = x_0^2 + y_0^2 + (x_0y_0)^2 > (x_0y_0)^2$$

so that $z_0 > x_0y_0$. Also $z_0 < y_0(x_0 + 1/x_0)$ because

$$z_0^2 < (x_0y_0)^2 + 2y_0^2 = y_0^2(x_0^2 + 2) < y_0^2(x_0^2 + 2 + 1/x_0^2).$$

Hence

$$(3) \qquad \begin{aligned} (2x_0^2 + 1)y_0 - 2x_0z_0 &< (2x_0^2 + 1)y_0 - 2x_0^2y_0 = y_0, \\ (2x_0^2 + 1)y_0 - 2x_0z_0 &> (2x_0^2 + 1)y_0 - 2x_0y_0(x_0 + 1/x_0) = -y_0. \end{aligned}$$

Let X, Y denote the transformations

$$X[x, y, z] \to [(2y^2 + 1)x + 2yz, y, 2y(y^2 + 1)x + (2y^2 + 1)z],$$
$$Y[x, y, z] \to [x, (2x^2 + 1)y + 2xz, 2x(x^2 + 1)y + (2x^2 + 1)z].$$

Now if $y_0 > x_0$, there is by (3) a finite integer p such that

$$Y^{-p}(x_0, y_0, z_0) \to (x_0, y_1, z_1)$$

where $|y_1| < x_0$. If $y_1 \neq 0$ there exists a further finite integer q such that

$$X^{-q}(x_0, |y_1|, z_0) \to (x_2, |y_1|, z_2)$$

where $|x_2| < |y_1|$. Therefore by a finite number of transformations any solution is reducible to a solution of one of the forms $(0, N, N)$, $(N, 0, N)$. Since every step in the process is reversible, the conclusion follows.

The result of the repeated transformation $Y^n(x, y_0, z_0) \to (x, y_n, z_n)$ may, by a familiar device, be expressed in the form

$$y_n = \frac{y_0}{2}(v_1^{2n} + v_2^{2n}) + \frac{z_0}{2\sqrt{x^2+1}}(v_1^{2n} - v_2^{2n}),$$

$$z_n = \frac{z_0}{2}(v_1^{2n} + v_2^{2n}) + \frac{y_0\sqrt{x^2+1}}{2}(v_1^{2n} - v_2^{2n}),$$

where $v_1 = x + (x^2+1)^{1/2}$ and $v_2 = x - (x^2+1)^{1/2}$. As the transformations X, Y are not commutative, no simple expression for the result of a random sequence of transformations is possible.

Also solved partially by Norman Anning, Leon Bankoff, Daniel Finkel, Arthur Gregory, M. R. Kenner, R. C. Lyness, Joseph Rosenbaum, K. Subbarao, R. Venkatachalam Iyer, and the Proposer.

Editorial Note. The Diophantine equation treated above can be written in the form $(x^2+1)(y^2+1) = z^2+1$. This suggests the equation

$$(x^2 + a)(y^2 + b) = z^2 + c,$$

where no two of a, b, c, vanish. There need not be any solution, *e.g.*, $(x^2+1)(y^2-1) = z^2+1$; but if any solution, not $(0, 0, 0)$ exists, then, by reasoning similar to II above, there is an infinite number of solutions with x fixed, and an infinite number with y fixed.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, Oberlin College, Oberlin, Ohio, and not to any of the other editors or officers of the Association.

Elementary Analytic and Projective Geometry. By D. J. Struik. Cambridge, Addison-Wesley Press, Inc., 1953. 9+291 pages. $6.50.

Projective geometry may be developed either projectively or metrically. In the former case, no metric idea is used so that a projective mentality is estab-

lished without Euclidean prejudices. In the latter case, the concepts are approached from the metric side so that the treatment can be based on students' previous training in metric geometry. This book belongs to the second category.

The avowed purpose of the book is "to find the legitimate place for this field inside our present mathematical curriculum and to stress those fundamentals which are most vital for the understanding of our science as a whole." It is to be covered in one semester and only enough knowledge of elementary analytic geometry to recognize conics and quadrics is presupposed. Algebraic methods are occasionally exchanged for synthetic ones. The book is built around the Erlanger Program and contains well selected materials. Of the well known texts on this subject perhaps it most nearly parallels Graustein's Higher Geometry in subject matter and in method. But it enlarges the horizon by the introduction of historical remarks and connections with other fields while being only a little over half as long—an advantage for those colleges whose curriculum is so crowded that only a one-term course can be allotted to analytic geometry beyond the traditional course. It is certainly a very suitable and long-awaited text.

There are eleven chapters in the book. The first seven chapters cover the projective geometry of the plane and the last four chapters cover that of space. The titles of the chapters are: (1) Point Sets on a Line; (2) Line Pencils; (3) Line Coordinates. Homogeneous Coordinates; (4) Transformations of the Plane; (5) Projective Theory of Conics; (6) Affine and Euclidean Theory of Conics; (7) Projective Metric; (8) Points, Lines, and Planes; (9) Projective Theory of Quadrics; (10) Affine and Euclidean Theory of Quadrics; (11) Transformations of Space. Cross ratio is defined in terms of division ratios which are expressed in terms of Cartesian coordinates of points. Projectivity is defined as one-one algebraic correspondence between division ratios. Projective coordinates of a point are defined in terms of its trilinear coordinates and those of a line by means of its equation in projective point coordinates, which are shown to be expressible as cross ratios. That the projective plane is nonorientable and closed is illustrated with great clearness by its topologically equivalent figures. Einstein's summation convention is used to simplify the notation for projective transformation. Transformations of projective coordinates are called projective transformations when they are passive or collineations when they are active. Elementary divisors are used for the classification of collineation. A conic is defined as the locus of intersections of corresponding lines in two projective pencils. The concepts of point, line and plane in space are developed without metric ideas. Several sections are devoted to non-Euclidean geometries, vectors, force systems, central projection, perspective, parallel projection, oblique projection, and orthogonal projection.

The book is provided with collateral reading, with answers to exercises and with an index. The exercises are well selected, and the special exercises at the end of the book are most interesting.

There are several inconsistencies with regard to symbols and terminology

used in the book. On page 3, it is unfortunate but convenient to assign "the values $\lambda = \pm \infty$" to a point in order to establish correspondence between division ratio and point on a line. On page 110, one finds the statement "the value $\pm \infty$." On pp. 9, 11, 15, 23, 24, 55, the symbol ∞ is used instead of $\pm \infty$. Similarly, for a slope, $\pm \infty$ is used on p. 27 and ∞ is used on p. 136. With respect to Ex. 1 on p. 6, cross-ratio of p. 8, and the statement about the bilinear relation (5-1) on p. 10, the symbol $\pm \infty$ should be assumed as one value and hence ∞ seems a better symbol for the division ratio of that certain point B on page 3. Operations with this symbol should be specifically indicated, since (5-1) is taken as the definition of projectivity and plays an important role in its following development.

On page 3 the author decides to use "ideal versus ordinary." But on pp. 12, 59, 138, 142, 185, 210, the terms "at infinity," "improper," and "special" are adopted instead of "ideal." This use of different terms for the same kind of object on different pages might cause confusion to the beginner. Similar cases happen on p. 14 and p. 15 concerning the terms "standard form" and "canonical form" and on p. 224 with respect to "quartic" and "biquadratic."

On page 11, after proving the theorem that the cross ratio of four points on a line is invariant under a (non-singular) projectivity, the author states that "we return to this theorem in Chapter 2," which is found on p. 42 and reads as "the cross ratio is invariant under projection." The term "projection" is not defined. From the figures on the same page, it seems to have the general geometrical meaning as described on p. 232. Although the equivalence between projection and projectivity, which is defined algebraically, follows immediately from a theorem on p. 24, it seems to the reviewer an explanation might be helpful to convey to the reader a clear geometrical concept of projectivity.

The equation of a plane in geometry of space is usually understood to be a shorthand statement for "the equation connecting the coordinates of the points of a plane." Such a term as "the equation of a plane in plane coordinates" is misleading and it is unfortunately used for coordinates of a plane on p. 178 and p. 198.

There are many typographical errors in the book. Though most of the errors are obvious, yet they do cause a lot of trouble to the reader, especially to the beginner. It is the reviewer's hope that a list of errata be immediately set up and sold with the book.

T. K. PAN
University of Oklahoma

Computing Manual. Third Edition. By Fred Gruenberger, Madison, University of Wisconsin Press, 1952. 123 pages.

The title of this small paper-bound manual is misleading. Opposed to the broadness the title suggests, it is devoted exclusively to technical problems

arising in the use of standard I.B.M. punched card equipment for computing, and was intended as an aid in training personnel for such work.

Most of the chapters of the manual are devoted to the handling of some familiar computing problem. Digiting and differencing with a tabulator, and chi-squared analysis on a calculating punch, 602A, are some of the topics discussed. There is also a short section on the layout of questionnaires, keeping in mind that I.B.M. punched cards are going to be used in the analysis. In keeping with its practical intent, throughout the book, emphasis is placed on ease of handling, expense and error detection. The last quarter of the book has various laboratory problems and exercises.

Although there is a cursory discussion of the card programmed electronic Calculator, C.P.C. model one, there is no mention of the more recent I.B.M. machines such as the 101, 604, 701, *etc.*

Because the author assumes a working knowledge of the machines discussed and of the associated I.B.M. terminology, one unfamiliar with these would find little to interest him. On the other hand, an experienced operator would most likely be familiar with the contents. Although the manual was used as a text in a course on computing machines at Wisconsin, I feel that it is too technical to interest those students not going directly into I.B.M. computing. As was said, the manual was designed to aid the instruction of personnel in I.B.M. computing and, as such, should prove very useful.

GLENN LEWIS
Computer Project
Institute for Advanced Study

Real Functions. By Casper Goffman. New York, Rinehart and Company, Inc. 1953. 12+263 pages. $6.00.

This is a text-book based on a course in functions of a real variable given by the author to first-year graduate students at the University of Oklahoma. It is intended for a year course meeting three times per week. The material covered is indicated by the following Chapter headings: 1. Sets and Operations. 2. Equivalence of Sets. 3. The Real Numbers. 4. Limit Theorems. 5. Simple Properties of Sets. 6. The Cantor Ternary Set. 7. Functions. 8. Sequences of Functions. 9. The Derivative of a Function. 10. Order Types and Ordinals. 11. Borel Sets and Baire Functions. 12. Applications of Well-Ordering. 13. Measure and Measurable Sets. 14. Metric Properties of Sets. 15. Measurable Functions. 16. Approximation of Measurable Functions. 17. The Lebesgue Integral and the Riemann Integral. 18. The Lebesgue Integral as a Set Function. 19. The Fundamental Theorem of the Calculus. 20. Planar Measure and Double Integration.

The book is carefully and clearly written, being directed primarily to the student who is meeting the subject for the first time rather than to the expert. Especially in the earlier chapters, the author patiently leads the reader over

those pitfalls which are so obvious in hindsight but give students so much trouble when first encountered. There are numerous examples and exercises. The exercises vary greatly in difficulty from rather trivial remarks on the one hand to some really difficult problems on the other which will tax the most mature students. In the reviewer's opinion, the exercises would be considerably more useful if there were some indications of difficulty with more hints and cross-references to the text, other exercises or outside reading.

In Chapter 3 the real numbers are constructed using the rationals (defined as an ordered field) as a starting point. Both the Cantor and Dedekind methods for completing the rationals are outlined. However it should be pointed out that the author gives an inadequate proof of completeness for the real numbers so obtained. What is missing is a proof that his definitions of fundamental sequence and limit of a sequence of real numbers are equivalent to the usual definitions in terms of absolute values.

Slightly less than half the book is devoted to measure theory and Lebesgue integration. The theory is developed for sets on the unit intervals (except for Chapter 20). Unbounded sets are avoided as a rule except in a number of exercises where the student is asked to carry out a discussion for the general case. The last chapter contains a proof of the Fubini theorem.

This book should prove to be very useful either as a text-book or as a reference book for a first course in real variables. The average beginning graduate student with a reasonably good background should be able to read the book on his own with very little trouble except possibly with the more difficult exercises.

C. E. RICKART
Yale University

NEW BOOKS RECEIVED

Mechanics. By K. R. Symon. Cambridge, Mass., Addison-Wesley Press, Inc., 1953. 13+358 pages. $7.50.

Sample Survey Methods and Theory. Volume I. By M. H. Hansen, W. N. Hurwitz, and W. G. Madow. New York, John Wiley & Sons, Inc., 1953. 22+638 pages. $8.00.

An Introduction to the History of Mathematics. By Howard Eves. New York, Rinehart & Company, Inc., 1953. 15+422 pages. $6.00.

Methods of Mathematical Physics, Volume I. By R. Courant and D. Hilbert. New York, Interscience Publishers, 1953. 15+571 pages. $9.50.

Mathematical Research in India. By A. C. Banerji. Lahore, India. The Indian Science Congress Association and the Council of Scientific & Industrial Research, 1953. 43 pages.

Graphs of the Compton Energy-Angle Relationship and the Klein-Nishina Formula from 10 Kev to 500 Mev. By Ann T. Nelms. National Bureau of Standards Circular 542. 89 pages. $0.55.

Original Investigation or How to Attack an Exercise in Geometry. By E. S. Loomis. Columbus, Bonded Scale and Machine Co., 1953. No charge.

Introduction to Dynamics. By L. A. Pars. New York, Cambridge University Press, 1953. 22+501 pages. $6.00.

An Introduction to Homotopy Theory. By P. J. Hilton. New York, Cambridge University Press, 1953. 8+142 pages. $3.00.

An Introduction to Relaxation Methods. By F. S. Shaw. New York, Dover Publishing Company, 1953, 396 pages. $5.50.

Partial Differential Equations in Engineering Problems. By K. S. Miller. New York, Prentice-Hall, Inc., 1953. 8+254 pages. $4.75.

Plane Trigonometry. Second Edition. By A. W. Weeks and H. G. Funkhouser. New York, D. Van Nostrand Company, 1953. 8+197+37 pages. $2.68 (without tables), $2.88 (with tables).

A Mathematician's Miscellany. By J. E. Littlewood. London, Methuen and Company, 1953. 7+136 pages. 15s.

Plane Trigonometry. By P. R. Rider. New York, The Macmillan Company, 1953. 8+180 pages. $3.00.

Differential Line Geometry. By Vaclav Hlavaty. Translation based on text by H. Levy. Groningen, Holland, P. Noordhoff Ltd., 1953. 10+495 pages. f 25.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

NATIONAL REGISTER OF SCIENTIFIC PERSONNEL

The National Science Foundation is organizing a National Register of Scientific and Technical Personnel. Unlike the Roster compiled during World War II and the Register of 1950–51, this one will be a continuing affair: information will be kept up to date.

The Register is organized in sections. The American Mathematical Society has agreed to compile and maintain the section of the Register covering the mathematical sciences.

There are two major uses to which this Register can be put. In case of a national emergency the Register will quickly provide lists of specialists to whom essential work might be referred by panels of experts chosen by the appropriate scientific organizations.

The second use of the Register is in connection with statistical studies of scientific manpower. There is no doubt that the country's demand for scientific manpower far exceeds the supply. Effective policies for increasing this supply must be based on accurate information about scientists. Such information is not now available but can be furnished to a considerable extent by this Register.

Both these uses of this Register are highly important, and so you are urged to complete and return the questionnaire which you will receive soon.

SYMPOSIUM ON MONTE CARLO METHODS

Preliminary Announcement: A Symposium on Monte Carlo Methods, sponsored by the Aeronautical Research Laboratory, Wright Air Development Center, will be conducted by the Statistical Laboratory, University of Florida, at Gainesville on March 16 and 17, 1954. Registration will be on Monday, March 15, for those who arrive early. An invitation is issued to those interested in the field to attend. Further information may be obtained by writing Professor H. A. Meyer, Building OE, University of Florida, Gainesville.

Following the Symposium, an Eastern Regional meeting of the Institute of Mathematical Statistics is being planned for Thursday, March 18, 1954, at Gainesville. The Biometric Society, ENAR is meeting on March 18, 19 and 20.

PERSONAL ITEMS

Dr. Morris Ostrofsky of the Westinghouse Research Laboratories, Pittsburgh, represented the Association at the Diamond Jubilee Ceremonies celebrating the founding of Duquesne University on November 11–12, 1953.

Dean H. R. Kingston of the University of Western Ontario received an honorary degree of Doctor of Laws from Queen's University in June, 1953.

Dr. C. V. Newsom, Associate Commissioner of Higher Education in the State of New York, was granted an honorary degree of Doctor of Humane Letters by Hofstra College in June, 1953.

Professor G. deB. Robinson of the University of Toronto has been elected President of the Canadian Mathematical Congress for 1953–57.

Boston University announces the following: Dr. D. W. Blackett, formerly research associate at Princeton University, has been appointed to an assistant professorship; Dr. F. E. Browder is now a Guggenheim Fellow and is at the Institute for Advanced Study; Dr. V. R. Staknis has been appointed to an assistant professorship at Northeastern University.

Brooklyn College reports the following: Professor Walter Prenowitz is the recipient of a Ford Foundation Fellowship for the year 1953–54 and is at the Institute for Advanced Study; Dr. William Forman has been appointed to an instructorship; Mr. Leon Herbach has resigned and is with the Army Chemical Corps at New York University.

At California Institute of Technology: Dr. A. G. Mackie has been appointed Bateman Fellow; Dr. Frank Spitzer has been appointed to an instructorship.

Emory University announces the following: Dr. Morris Friedman, previously acting assistant professor at Tulane University, has been appointed to an assistant professorship; Associate Professor C. E. Clark has resigned.

Hampton Institute reports the following: Mr. S. R. Beyma, previously with the United States Army, has been appointed to an instructorship; Associate Professor C. Y. Wang is now an instructor at the University of Minnesota.

Illinois Institute of Technology announces: Mr. George Crane, Mr. Thomas Erber, and Mr. Bernard Galler have been appointed to part-time instructorships; Associate Professor Michael Sadowsky has been appointed Professor of Mechanics at Rensselaer Polytechnic Institute.

Iowa State College makes the following announcements: Dr. G. W. Peglar has been promoted to an assistant professorship; Dr. W. D. Lindstrom, previously at the State University of Iowa, Dr. D. E. Sanderson, formerly at the University of Wisconsin, and Dr. F. M. Wright of Northwestern University have been appointed to instructorships; Dr. H. J. Weiss, formerly located at Brown University, has been appointed to an assistant professorship.

Marquette University reports the following: Dr. C. B. Hanneken, formerly an assistant at the University of Illinois, and Mr. William Golomski of St. Louis University have been appointed to instructorships; Associate Professor B. F. Dostal, recently retired from the University of Florida, has been appointed a part-time lecturer; Associate Professor L. V. Toralballa is now at New York University.

Ohio State University announces: Dr. Erwin Kleinfeld, previously an instructor at the University of Chicago, has been appointed to an assistant professorship; Dr. D. W. Dubois of the University of Oklahoma has been appointed to an instructorship.

Oregon State College announces the following: Mr. R. L. Brock, who is on leave from Boeing Airplane Company, Seattle, Washington, has been appointed Acting Instructor; Assistant Professor W. M. Stone, who has returned after a two-year leave of absence spent at Boeing Airplane Company has been promoted to an associate professorship.

Rensselaer Polytechnic Institute makes the following announcements: Dr. Kurt Bing of the University of California has been appointed to an assistant professorship; Associate Professor Horace Komm of the University of the South and Instructor Valdemars Punga of the University of Massachusetts have also been appointed to assistant professorships; Mr. W. R. Beck has resigned and is at White Plains High School; Professor Dis Maly has been elected a member of the Council of the Association of Mathematics Teachers of New York State.

At St. Louis University: Reverend J. F. Daly of Rockhurst College and Mr. W. E. Perrault of Fairfield University have been appointed to instructorships; Instructor Paolo Lanzano has been promoted to an assistant professorship;

Assistant Professor A. J. Lorenz has resigned from his teaching duties and is doing graduate work at the University.

Syracuse University reports the following: Dr. K. L. Chung, previously visiting associate professor at Cornell University, has been appointed to an associate professorship; Associate Professor R. D. Whitney has retired; Assistant Professor Erik Hemmingsen has received an award from the Danish National Science Foundation.

State College of Washington announces: Professor S. G. Hacker has been appointed Professor of Mathematics and Director of the J. R. Jewett Observatory; Assistant Professor J. L. Brenner has been promoted to an associate professorship; Dr. P. A. Clement has been promoted to an assistant professorship; Dr. W. J. Firey of Stanford University, Dr. W. M. Gilbert of the Analytical Research Group of Princeton University, and Dr. M. T. Wechsler of Princeton University have been appointed to instructorships.

Texas Southern University reports the following: Professor J. A. Pierce, head of the Department of Mathematics and chairman of the Division of Natural Sciences and Mathematics, has been appointed Dean of the Graduate School; Miss Thyrsa A. Frazier, previously an analytical statistician at the Wright-Patterson Air Force Base, has been appointed to an instructorship; Mr. John Westberry and Mr. A. H. Wardlow have received renewals of their General Education Board Fellowships for study at the University of Michigan.

At the University of Alabama: Professor M. O. Gonzalez of the University of Havana has been appointed to a professorship; Dr. H. C. Filgo, previously a graduate assistant at Rice Institute, Instructor C. C. Buck of Wayne University, Mr. J. W. Jewett, formerly a teaching fellow at the University of Michigan, and Mr. B. M. Seelbinder, previously a part-time instructor at the University of North Carolina, have been appointed to assistant professorships; Mrs. Margaret B. Seelbinder, formerly a part-time instructor at the University of North Carolina, has been appointed to an instructorship.

University of California at Berkeley announces: Professors D. H. Lehmer and Hans Lewy have returned to their positions in the Department; Associate Professor J. L. Kelley, who has been promoted to a professorship, is on sabbatical leave for the year 1953–54 and holds a National Science Foundation grant; Professor Harald Cramer, who is also Rector of the University of Stockholm, served as Visiting Professor during the fall term; Professor Jan van der Corput is Visiting Professor during the year; Associate Professor Leon Henkin of the University of Southern California has been appointed to a professorship; Professor Herman Meyer of the College of the University of Chicago is Visiting Associate Professor for the year; Dr. Louise C. Lim has been awarded a Ford Foundation Fellowship and was at the University during the fall term; Associate Professor A. Seidenberg is on sabbatical leave for the year 1953–54 and is at Cambridge, Massachusetts; Professor L. H. Seinford is on leave of absence because of illness; Dr. Anne C. Davis, Dr. M. P. Epstein, Dr. D. C. Kleinecke, Dr. J. P. Roth, and Dr. R. A. Wijsman have been appointed to instructorships; Dr.

Joseph Putter has been appointed Lecturer in the Statistical Laboratory.

University of California at Los Angeles reports the following: Dr. J. R. Jackson, previously a research assistant at the University, has been appointed to an instructorship; Dr. Maurice Sion, formerly an instructor at the University of California at Berkeley, has been appointed to an instructorship; Assistant Professor P. G. Hodge, Jr., has been promoted to an associate professorship; during the year 1953–54, Professor Hodge is on leave and holds the position of Associate Professor of Applied Mathematics at Polytechnic Institute of Brooklyn; Associate Professor Richard Arens is on sabbatical leave and is at the Institute for Advanced Study; Professor P. G. Hoel is also on sabbatical leave and is in Norway on a Fulbright Fellowship; Assistant Professor L. J. Paige is on sabbatical leave and is at the Institute for Advanced Study.

University of Colorado announces: Dr. E. B. McLeod, previously a research assistant at Stanford University, and Dr. A. Zirakzadeh, who has been an instructor at Oklahoma Agricultural and Mechanical College, have been appointed to instructorships; Dr. Robert Osserman has resigned.

University of Connecticut announces the following appointments to instructorships: Dr. Oscar Litoff, Mr. A. E. Nussbaum, Dr. E. E. Osborne, and Mr. John Rausen.

University of Delaware reports: Assistant Professor Russell Remage, Jr., has been promoted to an associate professorship; Dr. E. J. Pellicciaro, previously a graduate fellow at the University of North Carolina, has been appointed to an instructorship; Assistant Professor G. O. Peters has resigned.

At the University of Florida: Associate Professor E. H. Hadlock has been promoted to a professorship; Mr. R. W. Bagley, previously a graduate assistant at the University, Miss Elaine Hundertmark, formerly a teaching assistant at the University of Illinois, and Mr. J. D. Neff, member of the Technical Staff, Bell Telephone Laboratories, New York City, have been appointed to instructorships; Associate Professor B. F. Dostal has retired; Mr. L. W. Blanton is on leave of absence to continue his graduate study at Columbia University.

University of Houston makes the following announcements: Assistant Professors Blanche Grover, Ruth Kissel, and C. A. Rogers have been promoted to associate professorships; Dr. J. C. Douglas, Jr., Mrs. Thelma I. Hammerling, Miss Eleanor S. Mohr, Mr. W. R. Strickler, and Mr. R. O. Young have been appointed part-time instructorships; Mr. W. S. Rees has returned to the University as an assistant professor; Mr. D. O. Gray has returned from military leave.

University of Idaho announces: Assistant Professor A. E. Halteman has been promoted to an associate professorship; Mr. G. E. Witter has been promoted to an instructorship; Mr. P. E. Livermore and Mr. R. G. Schrandt, who have been discharged from military service recently, have been appointed to instructorships; Mr. R. J. Rohlfs, previously a graduate student at the University of South Dakota, has been appointed to an instructorship.

University of Kentucky announces the following: Professor M. C. Brown

has been appointed Acting Head of the Department; Mr. Howard Burnette, Miss Sara Ripy, Mr. Richard Sprague, Mr. W. C. Swift, Mr. Sherman Vanaman, Mr. J. B. Wells, and Mr. Wilson Zaring have been promoted to full-time instructorships; Mr. Cephas Bevins and Mr. Thomas Rowland have been appointed to part-time instructorships; Dr. Cordell Moore has resigned and has accepted a position with Convair Aircraft Corporation, Fort Worth, Texas; Mrs. Virginia B. Leach has resigned.

The University of Maine announces the appointments of Mr. J. S. Dinsmore, previously a graduate student at the University of North Carolina, and Assistant Professor D. A. Kearns of Merrimack College, Andover, Massachusetts, to instructorships.

At the University of Massachusetts: Mr. August Newlander, who has been a staff research mathematician at the Denver Research Institute, and Mr. C. W. Naylor, formerly master at the Darrow School, New Lebanon, New York, have been appointed to instructorships; Miss Lorraine D. Lavallee, previously a student at Mount Holyoke College, Mr. E. I. Pina and Mrs. Louise E. Rice, formerly students at the University, have been appointed to part-time instructorships; Mr. R. C. Scott has been appointed to a teaching fellowship; Mr. A. A. Kheiralla has resigned.

The University of Miami announces the following appointments: Dr. Charles Capel of Tulane University and Dr. R. A. Roberts of the University of West Virginia to assistant professorships; Professor E. J. Moulton of Northwestern University to a visiting professorship.

University of Mississippi reports the appointments of Mr. Russell Stokes and Mrs. Fred White to instructorships and the retirement of Assistant Professor A. H. Samuels.

University of New Mexico makes the following announcements: Dr. J. V. Lewis, previously at the Ballistic Research Laboratory, Aberdeen Proving Ground, has been appointed to an associate professorship; Dr. Oswald Wyler, formerly a lecturer at Northwestern University, has been appointed to an assistant professorship; Mr. Milton Hoehn of the University of Idaho and Mr. Arthur Steger, previously a graduate assistant at the University of California, have been appointed to instructorships; Professor M. S. Hendrickson has been elected Chairman of the Southwestern section of the Association for the year 1953–54.

University of North Dakota announces the following: Instructor W. H. McBride has been promoted to an assistant professorship; Associate Professor W. J. Lyche of Augustana College and Mr. K. H. Olson, previously a teaching assistant at the University, have been appointed to instructorships.

University of Oklahoma reports: Assistant Professor Earl LaFon has been promoted to an associate professorship; Dr. T. K. Pan, formerly visiting assistant professor at the University of California, has been appointed Visiting Associate Professor; Dr. J. B. Giever of the Instrumentation Laboratory, Massachusetts Institute of Technology, has been appointed to an assistant

professorship; Associate Professor Andrew Sobczyk has resigned and has accepted a position as a mathematician at Los Alamos Scientific Laboratory.

University of Oregon announces: Dr. S. G. Ghurye, previously a research associate at the University of North Carolina, and Assistant Professor Bertram Yood of Cornell University, have been appointed to assistant professorships; Dr. R. L. San Soucie, formerly a National Science Foundation Fellow at the University of Wisconsin, has been appointed to an instructorship; Dr. H. J. Reiter has returned to Austria; Assistant Professor F. J. Massey has been promoted to an associate professorship; Professor Massey is on leave and is a Ford Foundation Fellow at Harvard University.

University of Rochester reports the following: Assistant Professor H. P. Adkins has been promoted to an associate professorship and also holds the position of Assistant to the Dean of Men; Mrs. Mary Estill Rudin, of Duke University has been appointed to a part-time assistant professorship; Assistant Professor D. S. Miller has resigned and is with the Eastman Kodak Company.

At the University of Tennessee: Dr. Hanan Rubin, who is on leave from the Institute of Mathematical Sciences, New York University, has been appointed Visiting Assistant Professor; Dr. H. C. Griffith has been promoted to an instructorship; Professor Wallace Givens is on leave and is at the Computing Center of New York University.

University of Toronto announces the following: Professor L. J. Mordell, who has retired from the Sadlerian Professorship of Pure Mathematics in the University of Cambridge, is Visiting Professor for the Session 1953–54; Dr. P. C. Gilmore, who was awarded one of the Postdoctorate Fellowships of the National Research Council of Canada, is in residence as Research Associate; Assistant Professors Cecilia C. Krieger and Dr. D. A. S. Fraser have been promoted to associate professorships; Mr. W. O. J. Moser has been appointed to an instructorship; Dr. T. J. Jenkins, Mr. R. F. Johnston, Mr. E. E. Noonan, Mr. W. A. Skirrow, and Mr. A. W. Walker have been appointed to assistantships; Mr. D. W. Allan, Mr. L. L. Campbell, Mr. I. Guttman, Mr. A. H. Lightstone, Mr. K. Okashimo, Mr. F. G. Robinson, Mr. D. A. Sprott, and Mr. G. Zyskind have been appointed to teaching fellowships; Dr. G. A. Dirac has been appointed Senior Lecturer at King's College, London.

University of Tulsa reports the following: Associate Professor E. A. Howard has been promoted to a professorship; Assistant Professor Sarah M. Burkhart has been promoted to an associate professorship; Dr. Kenneth Walters, previously a graduate assistant at the University of Florida, has been appointed to an instructorship.

University of Utah reports the following: Assistant Professors R. E. Chamberlain and J. H. Wolfe have been promoted to associate professorships; Dr. C. E. Burgess has been promoted to an assistant professorship; Dr. Miriam L. Dickman, Mr. Louis Barrett, previously graduate assistants at the University, and Mr. C. W. Thompson, formerly part-time instructor at the University of Colorado, have been appointed to instructorships; Associate Professor R. N.

Thomas has resigned and is at the Harvard College Observatory; Dr. E. N. Parker is now Assistant Professor of Physics at the University; Dr. Henry Hiz has resigned to accept an appointment at the University of Pennsylvania; Mr. LaMar Deverall is now engaged as a mathematician at the Dugway Proving Grounds; Mr. Marcus Peterson has accepted a position with Consolidated Aircraft Company, Fort Worth, Texas; Dr. Elmo Stewart is with the Bendix Corporation.

At the University of Western Ontario: Mr. Arthur Woods has been appointed to a professorship; Professor R. H. Cole has been appointed to the Council of the Canadian Mathematical Congress.

Washington University announces the following: Dr. K. S. Shih, previously a graduate student at the University of Illinois, has been appointed Visiting Assistant Professor; Dr. Anne E. Scheerer, formerly a graduate student at the University of Pennsylvania, has been appointed to an instructorship; Professor Walter Leighton is on partial leave of absence from the University and is serving temporarily as Chief of the Mathematics Division of the Office of Scientific Research, United States Air Force; Professor T. L. Downs has been appointed Vice-Chairman for the current academic year; Dr. H. M. Schaerf has been awarded a Ford Foundation Fellowship for the year 1953–54.

Wayne University announces: Dr. Fred Meyer, formerly a research chemist with Ethyl Corporation, Detroit, has been appointed to an instructorship; Professor H. D. Huskey is now Assistant Director of the Institute for Numerical Analysis, National Bureau of Standards, Los Angeles.

Associate Professor Floyd Bowling of Lincoln Memorial University has been promoted to a professorship.

Dr. R. D. Branstetter of Iowa State College is affiliated now with the Operations Analysis Research Group, Colorado Springs, Colorado.

Mr. M. R. Bryson of the University of Idaho has been appointed to an instructorship at Drake University.

Mr. L. C. Damsgard of Pasadena City College has retired.

Mr. A. G. Davis of the University of Massachusetts has been appointed to an instructorship at Clarkson College of Technology.

Assistant Professor E. A. Davis is on leave and is studying at the University of Chicago under a Ford Foundation grant.

Assistant Professor Mary P. Dolciani of Vassar College has been awarded a Ford Foundation Fellowship for the year 1953–54 and is spending the year as an Honorary Research Assistant at University College, University of London.

Mr. A. B. Finkelstein of Long Island University has been promoted to an assistant professorship.

Miss Gloria C. Ford of Virginia State College is teaching now at Morgan College, Baltimore, Maryland.

Dr. M. L. Juncosa of the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, has accepted a position as mathematician with the Rand Corporation, Santa Monica, California.

Mr. R. G. Kuller has been appointed to an instructorship at Dartmouth College.

Associate Professor Marguerite Lehr of Bryn Mawr College gave a television course entitled "Invitation to Mathematics" during the first semester of 1952–53.

Dr. R. D. Luce of Massachusetts Institute of Technology has been appointed Managing Director of the Behavioral Models Project, Bureau of Applied Social Research, Columbia University.

Dr. Emanuel Parzen of the University of California has a position as a research scientist at the Hudson Laboratories, Columbia University.

Dr. R. L. Plunkett of the University of Virginia has been appointed to an assistant professorship at Vanderbilt University.

Dr. Gustave Rabson of Purdue University has been appointed to an assistant professorship at Antioch College.

Dr. C. R. Rao of Presidency College, Calcutta, India, has been appointed Visiting Professor of Mathematical Statistics at the University of Illinois.

Miss Margaret Roston has been appointed to an assistant professorship at Hood College.

Miss Ruth R. Royer of Iowa State College has been appointed to an instructorship at Chico State Teachers College, California.

Professor J. P. Scholz of Lebanon Valley College has been appointed to a professorship at Western College.

Associate Professor S. R. Smith of the University of Wyoming has been promoted to a professorship.

Professor G. W. Spencely of Miami University has retired with the title of Professor Emeritus.

Professor P. W. Stoner, previously chairman of the Department of Mathematics and Astronomy at Pasadena City College, has been appointed to a professorship at Westmount College.

Professor Emeritus J. I. Tracey of Yale University has been appointed Distinguished Professor of Mathematics at Texas Christian University for the year 1953–54.

Dr. A. B. Willcox has been appointed to an instructorship at Amherst College.

Reverend R. B. Eiten, who was Associate Professor of Mathematics at the University of Detroit, died on October 6, 1953.

Associate Professor O. W. Irwin, formerly of the faculty of Brooklyn College, died on August 22, 1953.

Professor Emeritus J. H. McDonald of the University of California died on July 4, 1953.

Mr. R. H. Mason of the University of Florida died on July 24, 1953.

Dr. Joseph Rosenbaum of the Milford School, Connecticut, died on November 10, 1953; he was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 45 persons have been elected to membership by the Board of Governors on applications duly certified.

E. N. BRANDT, JR., Student, University of Oklahoma.

P. N. W. BURGOYNE, Student, McGill University.

L. N. CAPLAN, Student, Carnegie Institute of Technology.

G. E. CARUSO, Student, St. John's University.

D. H. CLANTON, M.A. (Baylor) Instr., University of South Carolina.

G. M. CLOUGH, B.S. (New Mexico Inst.) Mathematician, Battelle Memorial Institute, Columbus, Ohio.

P. E. DALBEC, Student, Boston College.

S. H. DALRYMPLE, Student, University of Texas.

A. E. DEAN, B.S. (U.S. Naval Post.) Grad. Student, University of Mississippi.

S. H. EISMAN, M.A. (Pennsylvania) Optometrist, 6751 Souder Street, Philadelphia 49, Pa.

J. F. ELLIOTT, M.S. (Tennessee) Teacher, Dobyns-Bennett High School, Kingsport, Tenn.

ARTHUR EVANS, JR., Student, Carnegie Institute of Technology.

A. B. FINKELSTEIN, Ph.D. (N.Y.U.) Asst. Professor, Long Island University.

C. E. FLANAGAN, Ph.M. (Wisconsin) Director, Academic Education Department, Wisconsin State College, Whitewater, Wis.

EDWARD HALAS, Student, University of Detroit.

C. B. HANNEKEN, Ph.D. (Illinois) Instr., Marquette University.

ROBERT KALIN, M.A. (Harvard) Test-Specialist, Educational Testing Service, Princeton, N. J.

H. E. KANTER, Student, Carnegie Institute of Technology.

LOLA F. KISER, B.S. (Memphis S.C.) Teaching Assistant, University of Georgia.

HORACE KOMM, Ph.D. (Michigan) Asst. Professor, Rensselaer Polytechnic Institute.

STEPHEN KULIK, Dr.Math. (Kiev) Asso. Professor, University of South Carolina.

R. G. KULLER, M.S. (Michigan) Instr., Dartmouth College.

R. F. LISKOVEC, B.S. (Kent State) 4126 East 131st Street, Cleveland 5, Ohio.

R. D. LUCE, Ph.D. (M.I.T.) Managing Director, Behavorial Models Project, Bureau of Applied Social Research, Columbia University.

MRS. HELEN M. MARSTON, M.A. (Duke) Head, Mathematics Section, Educational Testing Service, Princeton, N. J.

G. E. MEIKE, B.S. (Aquinas) Teaching Fellow, University of Detroit.

M. W. MILLIGAN, M.A. (Illinois) Asst. Professor, Adams State College, Alamosa, Colo.

M. A. OBERDICK, Student, Seton Hill University.

T. K. PAN, Ph.D. (California) Visiting Asso. Professor, University of Oklahoma.

R. N. PENDERGRASS, M.A. (Missouri) Asso. Professor, Radford College, Virginia.

J. H. POWELL, M.A. (Michigan S.C.) Grad. Research Assistant, Michigan State College.

VALDEMARS PUNGA, M.Math.Sci. (Latvia) Asst. Professor, Rensselaer Polytechnic Institute.

REV. J. H. RAYMOND, Ph.D. (U. of Washington) Chairman, St. Martin's College, Olympia, Wash.

MARGARET A. REIFF, M.S. (Oklahoma A. & M.) Temporary Instr., Oklahoma Agricultural and Mechanical College.

J. D. RICE, M.A. (Rice) Asst. Professor, Lamar State College of Technology, Beaumont, Tex.

D. A. ROBINSON, B.A. (New York S.T.C., Albany) Grad. Student, Rensselaer Polytechnic Institute.

P. G. ROONEY, Ph.D. (C.I.T.) Lecturer, University of Alberta.

R. S. ROTH, B.A. (Kenyon) Grad. Student, Carnegie Institute of Technology.

MIRIAM J. RUSSELL, M.A. (George Peabody) Instr., University of Arizona.

J. P. SCHOLZ, Ph.D. (Vienna) Acting Head, Department of Physics; Professor-elect of Mathematics, Western College.

MRS. OLIVIA H. SHANKS, M.A. (Vanderbilt) Chairman, Belmont College.

SISTER ANTONIETTA FITZPATRICK, M.A. (Texas) Instr., Incarnate Word College.

M. W. STONE, M.A. (George Peabody) Mathematician, Rohm and Haas Company, Redstone Arsenal, Huntsville, Ala.

D. J. VAN VRANKEN, B.S. (Union C.) Instr.; Grad. Student, Rensselaer Polytechnic Institute.

KENNETH WOLSSON, Student, Brooklyn College.

THE OCTOBER MEETING OF THE MINNESOTA SECTION

The October meeting of the Minnesota Section of the Mathematical Association of America was held at the Bemidji State Teachers College in Bemidji, Minnesota, on October 10, 1953. Sessions were held in the forenoon, at luncheon and in the afternoon. Professors H. D. Colson, R. C. Staley and A. G. Hill, Chairman of the Section, presided at the respective sessions.

Twenty-nine persons attended the meeting including the following twenty members of the Association:

F. J. Arena, J. M. Calloway, E. J. Camp, H. D. Colson, Ruby M. Grimes, F. C. Hatfield, A. G. Hill, Karlis Kaufmanis, W. H. McBride, Margaret Owchar, J. C. Peterson, P. A. Rognlie, L. W. Sheridan, F. C. Smith, R. C. Staley, O. E. Stanaitis, A. G. Swanson, Matilda B. Thompson, K. W. Wegner, F. L. Wolf.

By invitation of the Executive Committee, Professor O. E. Stanaitis of St. Olaf College delivered an address at the morning session entitled "Remarks on Abel's Partial Summation." Abstract of this address follows:

Abel's partial summation and the following more special criteria of convergence of series were reviewed: the test of Abel, Dirichlet, du Bois-Reymond and Dedekind, J. Hadamard, E. B. Eliot, and G. H. Hardy. It was shown that Euler's summation formula and a test superior to Hardy's test follow almost immediately from Abel's partial summation. Though the result is in any case a far-reaching one, the convergence of somewhat more general series cannot be proved, and additional refinement of the formulas obtained are necessary. The statements were illustrated by examples.

The following short papers were presented:

1. *The size of the visible universe*, by Professor Karlis Kaufmanis, Gustavus Adolphus College.

The author discussed the period-luminosity curve and pointed out the reasons that have caused the astronomers to multiply all distances determined by means of the classical cepheids by the factor of two. The maximum range of the 200-inch telescope on Mt. Palomar is $2 \cdot 10^9$ light years as estimated by Baade.

2. *P-adic representations of p-primary groups without elements of finite order*, by Professor G. C. Preston, Macalester College, introduced by Professor G. K. Kalisch.

Let G be a locally compact p-primary group* and $p^\infty = Q_p - z_p$ where Q_p represents the p-adic numbers and Z_p the p-adic integers. (G, p^∞) will represent the group of all continuous homomorphisms of G into p^∞ provided with the following topology: a neighborhood of zero will consist of all homomorphisms of (G, p^∞) which map some fixed compact set of G onto the zero element of p^∞. Then $(G, p^\infty) = \widehat{G}$ where $\widehat{G}$ is the ordinary Pontrjagin character group of G. Therefore, $((G, p^\infty), p^\infty) = \widehat{\widehat{G}} = G$. From this fact it is also clear that G has "sufficiently many" continuous homomorphisms into p^∞; *i.e.*, given $a \neq b$ in G, there is a continuous homomorphism α of G into p^∞ such that $\alpha(a) \neq \alpha(b)$.

Now if G' is a linear topological space over Q_p, the set of continuous homomorphisms of G' into Q_p (1-dimensional representations in Q_p) is algebraically isomorphic to (G, p^∞). Hence G' has sufficiently many 1-dimensional representations in Q_p. If G is a locally compact p-primary group without elements of finite order, there exists a linear topological space G' over Q_p such that $G \subset G'$ (Braconnier). Hence G has sufficiently many 1-dimensional representations in Q_p.

3. *Some remarks concerning 110 formulas for the area of a plane triangle*, by Professor F. J. Arena, North Dakota Agricultural College.

In this paper the writer discusses the various types of formulas for the area of a plane triangle published by Marcus Baker in the Annals of Mathematics, vol. 1, 1884, pp. 134–138 and vol. 2, 1885, pp. 11–18.

4. *Vector methods in analytical geometry*, by Professor W. L. Woodley, North Dakota Agricultural College, introduced by Professor A. G. Hill.

In recent years vector methods have been applied to analytics with considerable success.

A line segment is written in matrix form:

$$P_1P_2 = (x_2 - x_1 \quad y_2 - y_1 \quad z_2 - z_1) \quad \text{or} \quad l_1(\cos\alpha \quad \cos\beta \quad \cos\gamma).$$

In plane analytics, any two lines may be written:

$$L_1 = (a \quad b) = l_1(\cos\alpha_1 \quad \sin\alpha_1); \qquad L_2 = (c \quad d) = l_2(\cos\alpha_2 \quad \sin\alpha_2)$$

with the scalar product

$$L_1L_2 = ac + bd = l_1l_2(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2) = l_1l_2\cos\theta,$$

or

$$\cos\theta = \frac{L_1L_2}{l_1l_2}.$$

These formulas are readily extended to three or more dimensions. The perpendicularity condition is $L_1L_2 = 0$. The area of the triangle determined by L_1 and L_2 is $\pm\frac{1}{2}L_1'L_2$ where $L_1' = (b \quad -a)$ or $(-b \quad a)$ is perpendicular to L_1.

If four components are used, a world line may be written with time as a fourth component as follows:

$$(x - x_1 \quad y - y_1 \quad z - z_1 \quad t - t_1) = \phi(x_2 - x_1 \quad y_2 - y_1 \quad z_2 - z_1 \quad t_2 - t_1)$$

where ϕ is a parameter. This is a component equation. By equating components and eliminating ϕ,

* See J. Braconnier, Sur les groupes topologiques localement compacts, J. Math. Pures Appl., vol. 27, pp. 1–85, 1948; see also I. Kaplansky, Dual modules over a valuation rank, Proc. Amer. Math. Soc., vol. 4, pp. 213–219, 1953.

we may find the coordinate equations of the line. If ϕ is replaced by $r_1/(r_1+r_2)$, the segment P_1P_2 is divided in the ratio $r_1:r_2$.

A tangent to a curve may be written $(cdt\ idx\ idy\ idz)$ and has length $k_1ds = \sqrt{c^2dt^2-dx^2-dy^2-dz^2} = \sqrt{c^2-v^2}\ dt$. The solution of this equation has some interesting properties.

5. *Report on the summer conference in collegiate mathematics*, by Professor J. M. Calloway, Carleton College.

The purpose of the conference and the nature of the lectures were indicated as well as an evaluation of the successes and failures of the conference.

F. C. SMITH, *Secretary*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The Oklahoma Section of the Mathematical Association of America was the guest of Oklahoma City University, Oklahoma City, Oklahoma, on October 30, 1953. Professor W. N. Huff, Chairman of the Section, presided at the meeting. A total of eighty-four persons attended the meeting, including the following forty-two members of the Association:

R. V. Andree, Arthur Bernhart, P. M. Berry, J. C. Bradford, J. C. Brixey, Sarah M. Burkhart, R. B. Deal, R. C. Dragoo, J. R. Foote, I. E. Glover, E. V. Greer, L. D. Gregory, L. A. Guest, O. H. Hamilton, Claire A. Harrison, J. O. Hassler, E. F. Heimann, J. E. Hoffman, W. N. Huff, P. W. M. John, L. W. Johnson, Douglas Jones, J. E. LaFon, Gene Levy, B. L. Mackin, Dora McFarland, G. E. Meador, Dorothea Meagher, R. R. Murphy, F. J. Palas, D. L. Patten, G. M. Petersen, C. M. Pirrong, E. C. Rice, J. W. Sehestedt, M. G. Shults, O. S. Spears, C. E. Springer, Vivian Spurgeon, J. D. Thomas, G. R. Vick, J. H. Zant.

At the business meeting the following officers were elected: Chairman, Professor R. B. Deal, Oklahoma Agricultural and Mechanical College; Vice-Chairman, Professor C. M. Pirrong, Oklahoma City University; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma.

Mathematical literature, including the *O.U. Mathematics Letter*, was distributed after the business meeting.

The following papers were presented:

1. *Summation methods and continuity*, by Dr. G. M. Petersen, University of Oklahoma.

The usefulness of various methods of evaluating divergent series with relation to boundedness, circle of convergence of the related power series, *etc.*, was discussed. Examples of methods which evaluate rapidly diverging series, but which fail to sum bounded series, were given.

2. *Need for a dynamic theory of games*, by Professor Arthur Bernhart, University of Oklahoma.

The static theory of Von Neumann is not *relevant* to actual parlor game strategy. Several examples of *dynamic* situations where static theory needs modification were given.

3. *The Fregier point in euclidean n-space*, by Professor R. B. Deal, Oklahoma Agricultural and Mechanical College.

Orthogonal pairs of lines through a fixed point on a conic meet the conic in pairs of points of an orthogonal involution whose center lies on the lines joining corresponding points. This center is called the Fregier point for the given point and the conic. The locus of the Fregier point for a given conic is called the Fregier conic. The purpose of this investigation is to show that the concept generalized readily for orthogonal n-types of lines through a given point on a hyperquadric.

4. *The functional equation* $f^2(x)-f^2(y)=f(x+y)f(x-y)$, by Mr. J. E. Hoffman, University of Oklahoma, and Professor R. B. Deal, Oklahoma Agricultural and Mechanical College.

The only real continuous solutions of the functional equation $f^2(x)-f^2(y)=f(x+y)f(x-y)$ are found to be $f(x)=cx$, $f(x)=a \sinh bx$, and $f(x)=a \sin bx$. The methods, particularly those used in proving differentiability from continuity and the functional equation, are shown to be applicable to a number of other functional equations. Discontinuous solutions are shown to exist.

5. *Two typical elementary problems in artillery research*, by Major O. S. Spears, Fort Sill, Oklahoma.

In armed forces research, there are many important, interesting problems in elementary mathematics. The following are typical of certain problems which frequently arise in artillery research:

Problem 1: *The effect of a multiple-fuzing system on the time at which a projectile or warhead is expected to explode.* Ordinarily, the explosion of a projectile occurs when a pre-set timing mechanism (fuze) functions. If a normal distribution, $f(t)$, is assumed for a given type of fuze, then an interesting statistical problem arises when several identical fuzes are attached to the same projectile in such a manner that the projectile explodes when at least n of the fuzes function. (The variable t refers to the time at which a fuze functions.)

Problem 2: *Probability of artillery hits on aircraft flying along a battle front.* It is sometimes necessary for aircraft to fly parallel to a battle front in support of infantry, while friendly artillery is firing in a direction perpendicular to the same battle front, and in support of the same infantry. There exists, therefore, a certain probability that a given aircraft flying under stipulated conditions will be accidentally hit by an artillery shell. The magnitude of such probabilities can materially affect tactical doctrine. An approach to the problem is outlined.

6. *Hyperbolic non-euclidean geometry*, by Professor J. O. Hassler, University of Oklahoma. (By invitation.)

Professor Hassler substituted the Gauss-Bolyai-Lobachewski definition of parallel lines for the euclidean and the postulate that there are two parallels ("right-hand" and "left-hand") to a given line through an outside point for the Playfair form of the euclidean postulate of parallels, but included the other postulates and definitions of euclidean geometry in developing the usual sequence of theorems leading to the proof that the sum of the angles of a triangle is less than one straight angle.

The afternoon was devoted to a series of papers and panel discussions in conjunction with the Mathematics Section of the Oklahoma Education Association.

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CALENDAR OF FUTURE MEETINGS

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Marshall College, Huntington, West Virginia, May 1, 1954.

ILLINOIS, Knox College, Galesburg, May 14–15, 1954.

INDIANA, Rose Polytechnic Institute, Terre Haute, May, 1954.

IOWA, Iowa State College, Ames, April, 1954.

KANSAS, Baker University, Baldwin City, March 27, 1954.

KENTUCKY, April 24, 1954.

LOUISIANA-MISSISSIPPI, Tulane University, New Orleans, February 19–20, 1954.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, St. John's University, Brooklyn, March 27, 1953.

MICHIGAN, University of Michigan, Ann Arbor, March 27, 1954.

MINNESOTA, Hamline University, St. Paul, May 8, 1954.

MISSOURI, University of Missouri, Columbia, May 7, 1954.

NEBRASKA, Omaha, April 24, 1954.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 17, 1954.

OKLAHOMA, Oklahoma City University, October, 1954.

PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado Agricultural and Mechanical College, Fort Collins, April, 1954.

SOUTHEASTERN, University of South Carolina, Columbia, March 19–20, 1954.

SOUTHERN CALIFORNIA, George Pepperdine College, Los Angeles, March 13, 1954.

SOUTHWESTERN, Arizona State College, Tempe, April 16–17, 1954.

TEXAS, Texas Technological College, Lubbock, April 23–24, 1954.

UPPER NEW YORK STATE, College for Teachers at Albany, May 1, 1954.

WISCONSIN, State Teachers College, Eau Claire, May, 1954.

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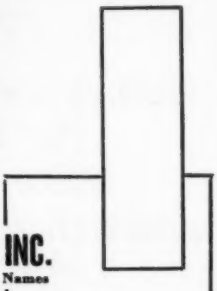

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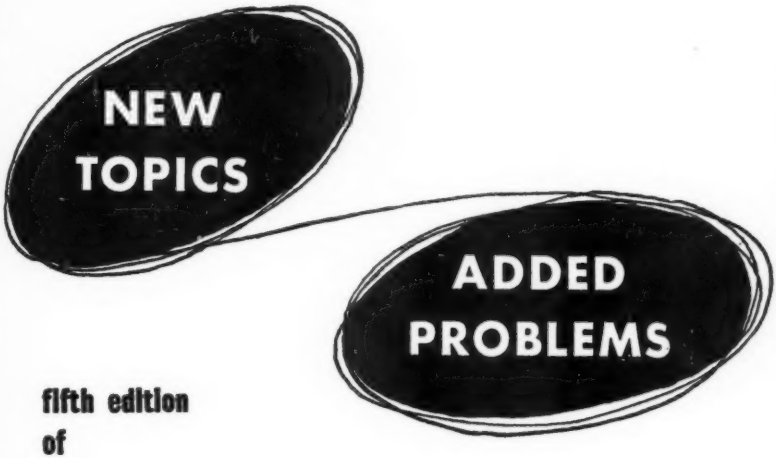

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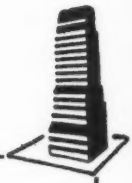

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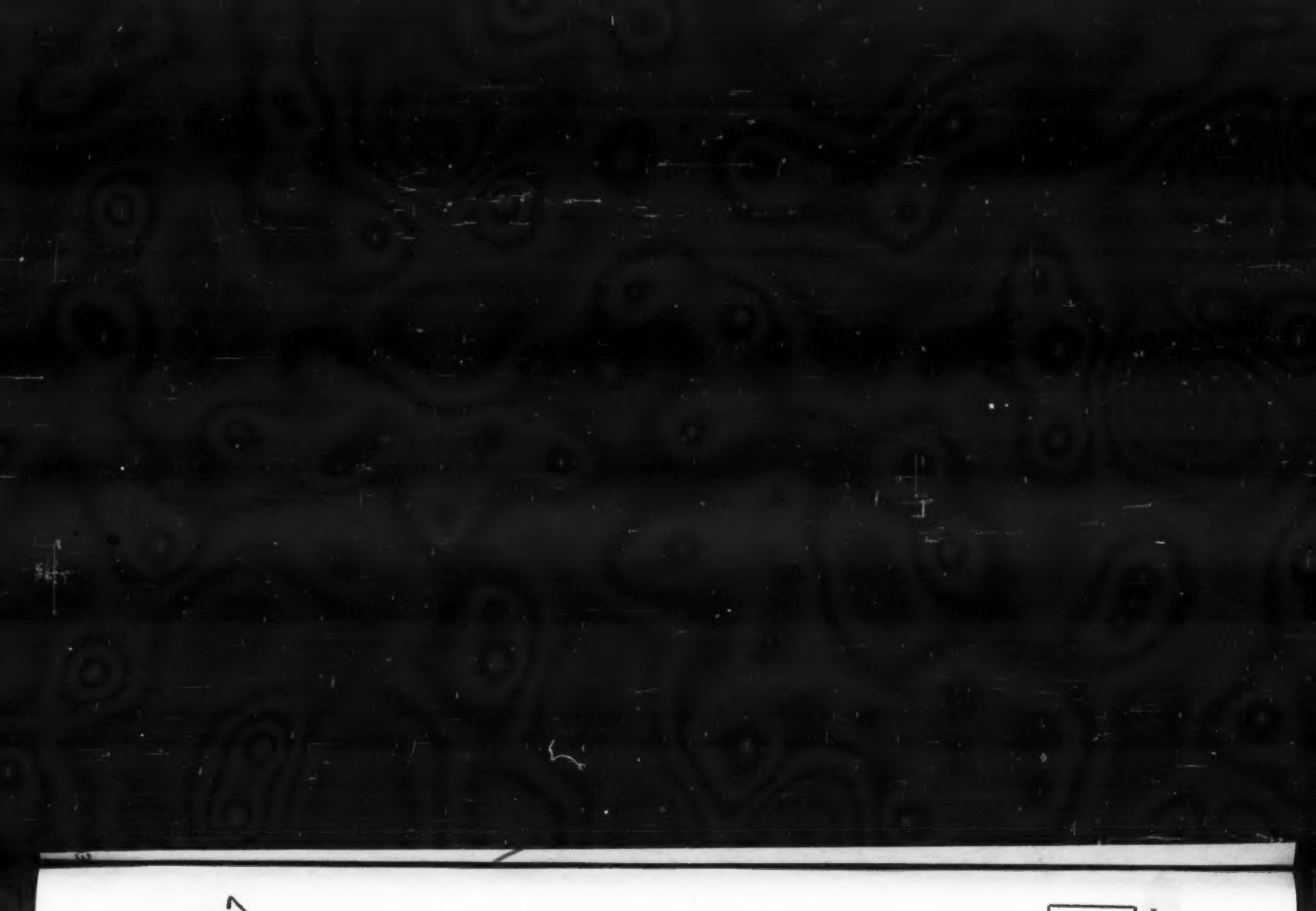
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